TOMITA-TAKESAKI THEORY

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1. Left and Right Hilbert Algebras

In this section we introduce the concept of a left and right Hilbert algebras, which we shall see has an important connection (as far as the dynamics of a von Neumann algebra are concerned) to weights, the topic of Section 3.

Definition 1.1. Let $\mathfrak{A}$ be an involutive algebra over $\mathbb{C}$ with involution $\xi \mapsto \xi^\sharp$ (resp. $\xi \mapsto \xi^\flat$). We say $\mathfrak{A}$ is a **left** (resp. **right**) **Hilbert algebra** if $\mathfrak{A}$ has a inner product $(\cdot \mid \cdot)$ satisfying:

a. Multiplication on the left (resp. right) is a bounded operator; that is, $\forall \xi \in \mathfrak{A}$ the map $\pi_l(\xi)\colon \eta \mapsto \xi\eta$ (resp. $\pi_r(\xi)\colon \eta \mapsto \eta\xi$) is bounded on $\mathfrak{A}$.
b. $(\xi\eta \mid \zeta) = (\eta \mid \xi^\sharp\zeta)$ (resp. $(\xi\eta \mid \zeta) = (\xi \mid \zeta\eta^\flat)$).
c. The involution map $\xi \mapsto \xi^\sharp$ (resp. $\xi \mapsto \xi^\flat$) is closable.
d. Denote by $\mathfrak{A}^2$ the linear span of products $\xi\eta$ for $\xi, \eta \in \mathfrak{A}$ (note this is a subalgebra). Then $\mathfrak{A}^2 \subset \mathfrak{A}$ is dense.

Suppose the involution map for a left Hilbert algebra $\mathfrak{A}$ is an (antilinear) isometry with respect to the inner product. We claim that $\mathfrak{A}$ is then a right Hilbert algebra as well with respect to the same involution. Indeed, we have

$$\|\pi_r(\xi)\eta\| = \|\eta\xi\| = \|(\eta\xi)^\sharp\| = \|\xi^\sharp\eta^\sharp\| = \|\pi_l(\xi^\sharp)\eta^\sharp\| \le \|\pi_l(\xi^\sharp)\|\|\eta^\sharp\| = \|\pi_l(\xi^\sharp)\|\|\eta\|,$$

(where the norm is the one derived from the inner product) ergo right multiplication is bounded. Moreover, if we define $\xi^\flat := \xi^\sharp$ then property (c) is immediate and (b) follows as well:

$$(\xi\eta \mid \zeta) = ((\xi\eta)^\sharp \mid \zeta^\sharp) = (\eta^\sharp\xi^\sharp \mid \zeta^\sharp) = (\xi^\sharp \mid \eta\zeta^\sharp) = (\xi \mid \zeta\eta^\sharp).$$

Thus $\mathfrak{A}$ is in fact also a right Hilbert algebra.

Propety (d) implies that the converse is true as well: suppose $\mathfrak{A}$ is a left Hilbert algebra such that the involution also makes it a right Hilbert algebra. Then

$$((\xi\eta)^\sharp \mid (\zeta\gamma)^\sharp) = ((\xi\eta)^\sharp \mid \gamma^\sharp\zeta^\sharp) = (\gamma(\xi\eta)^\sharp \mid \zeta^\sharp) = (\gamma \mid \zeta^\sharp\xi\eta) = (\zeta\gamma \mid \xi\eta),$$

ergo the involution is an antilinear isometry on the dense subalgebra $\mathfrak{A}^2$ and therefore is an antilinear isometry on $\mathfrak{A}$. This equilalence motivates the following definition.

Definition 1.2. A left Hilbert algebra $\mathfrak{A}$ whose involution is an antilinear isometry is called a **unimodular Hilbert algebra**, and we denote the involution by $\xi \mapsto \xi^*$.

Example 1.3. Let $\mathcal{M}$ be a von Neumann algebra with a faithful tracial state τ, then $\mathcal{M}$ is a unimodular Hilbert algebra with the same involution and inner product $(x \mid y) := \tau(y^*x)$.

Example 1.4. Let $\mathcal{M}$ be a von Neumann algebra and φ a positive linear functional. Let $\mathfrak{A}$ be the quotient of $\mathcal{M}$ by the subspace $\{x \in \mathcal{M}\colon \varphi(x^*x) = 0\}$ and η_φ the projection from $\mathcal{M}$ to $\mathfrak{A}$. Then $\mathfrak{A}$ is a left Hilbert algebra with involution $\eta_\varphi(x)^\sharp = \eta_\varphi(x^*)$ and inner product $(\eta_\varphi(x) \mid \eta_\varphi(y)) = \varphi(y^*x)$.

Example 1.5. Let G be a locally compact group with left Haar measure μ and recall the modular function $\delta_G(s)$ is defined to be the unique positive real number such that $\mu(\cdot s) = \delta_G(s)\mu(\cdot)$ (guaranteed by the uniqueness of the left Haar measure). Let $\mathcal{K}(G)$ be the space of continuous, compactly supported functions on G, then $\mathcal{K}(G)$ is a left Hilbert algebra with the following structure:

$$\begin{aligned}(\xi\eta)(s) &= \int_G \xi(t)\eta(t^{-1}s)\ d\mu(t);\\ \xi^\sharp(x) &= \delta_G(s^{-1})\overline{\xi(s^{-1})}, \qquad s\in G;\\ (\xi \mid \eta) &= \int_G \xi(s)\overline{\eta(s)}\ d\mu(s).\end{aligned}$$

We proceed with $\mathfrak{A}$ as a left Hilbert algebra. Let $\mathfrak{H}$ be the completion of $\mathfrak{A}$, then for each $\xi\in\mathfrak{A}$ we can extend $\pi_l(\xi)$ to $\mathfrak{H}$ so that $\pi_l(\xi)\in\mathcal{B}(\mathfrak{H})$ and π_l is a $*$-representation of $\mathfrak{A}$. Then density of $\mathfrak{A}^2$ implies that $\pi_l(\mathfrak{A})$ is non-degenerate.

Definition 1.6. The von Neumann algebra $\mathcal{R}_l(\mathfrak{A})=\pi_l(\mathfrak{A})''$ is called the **left von Neumann algebra**.
If $\mathfrak{A}$ is a right Hilbert algebra instead then we can define $\mathcal{R}_r(\mathfrak{A})=\pi_r(\mathfrak{A})''$.

One of intial goals is show that even when $\mathfrak{A}$ is merely a left Hilbert algebra we can associate to it a right Hilbert algebra $\mathfrak{A}'\subset\mathfrak{H}$ such that $\mathcal{R}_l(\mathfrak{A})'=\mathcal{R}_r(\mathfrak{A}')$. Consequently we will need to consider elements for which right multiplication is a bounded operation and will need to produce a new involution. Towards this end, we first need to study the involution more.

By property (c), the involution is closable and we denote its closure by S and the domain by $\mathfrak{D}^\sharp\subset\mathcal{H}$. So S is a densely defined unbounded operator with $\mathfrak{A}\subset\mathfrak{D}^\sharp$. We may continue to use the notation $\xi^\sharp=S\xi$ for $\xi\in\mathfrak{D}^\sharp$ even when $\xi\notin\mathfrak{A}$. Define a new inner product on $\mathfrak{D}^\sharp$:

$$(\xi\mid\eta)_\sharp := (\xi\mid\eta)+(S\eta\mid S\xi),$$

note the reversed order of ξ and η in the second term coming from the anti-linearity of S.

Lemma 1.7.

(i) *Let $\xi\in\mathfrak{H}$, then $\xi\in\mathfrak{D}^\sharp$ iff $\exists\{\xi_n\}\subset\mathfrak{A}$ a sequence such that $\lim_{n\to\infty}\|\xi_n-\xi\|=0$ and $\{\xi_n^\sharp\}$ is Cauchy in $\mathfrak{H}$. In which case we have*

$$\xi^\sharp=\lim_{n\to\infty}\xi_n^\sharp.$$

(ii) *$\mathfrak{D}^\sharp$ is complete with respect to the norm induced by $(\cdot\mid\cdot)_\sharp$ and $\mathfrak{A}$ is dense in $\mathfrak{D}^\sharp$.*

Proof.

(i): Suppose $\xi\in\mathfrak{D}^\sharp$, then $(\xi,S\xi)\subset\mathfrak{H}\oplus\mathfrak{H}$ is in the graph of S. Hence it is the limit of $\{(\xi_n,\xi_n^\sharp)\}\subset\mathfrak{A}\oplus\mathfrak{A}$ which implies $\lim_{n\to\infty}\|\xi_n-\xi\|=0$ and $\lim_{n\to\infty}\|\xi_n^\sharp-S\xi\|=0$; in particular, $\{\xi_n^\sharp\}$ is Cauchy.

Conversely, ξ is the norm limit of $\{\xi_n\}\subset\mathfrak{A}$, whose corresponding sharp sequence is Cauchy. But then $(\xi_n,\xi_n^\sharp)\subset\mathfrak{A}\oplus\mathfrak{A}$ is a Cauchy sequence in the graph of the involution map and therefore converges to a point on the graph of S. The condition of the sequence $\{\xi_n\}$ implies this limit point has ξ as its first coordinate and hence $\xi\in\mathfrak{D}^\sharp$ and

$$\xi^\sharp=\lim_{n\to\infty}\xi_n^\sharp.$$

(ii): Let $\{\xi_n\}$ be a Cauchy sequence with respect to the norm $\|\cdot\|_\sharp$ in $\mathfrak{D}^\sharp$. Since this norm dominates the original norm on $\mathfrak{H}$, we know this sequence is Cauchy with respect to the original norm and hence converges to some $\xi\in\mathfrak{H}$. We also know $\|\eta^\sharp\|\leq\|\eta\|_\sharp$ so that $\{\xi_n^\sharp\}$ is Cauchy as well. By part (i) we see that $\xi\in\mathfrak{D}^\sharp$ and hence $\mathfrak{D}^\sharp$ is complete with respect to this new norm.
The density of $\mathfrak{A}$ follows from its density in $\mathfrak{H}$ (and of course from the fact that $\mathfrak{A}\subset\mathfrak{D}^\sharp$). □

Lemma 1.8.

(i) $S=S^{-1}$.
(ii) *There exists an antilinear densely defined closed operator F with domain $\mathfrak{D}^\flat$ such that*
   a. $\mathfrak{D}^\flat=\{\eta\in\mathfrak{H}\colon \xi\in\mathfrak{D}^\sharp\mapsto(\eta\mid S\xi)$ *is bounded*$\}$;
   b. $(S\xi\mid\eta)=(F\eta\mid\xi),\qquad \xi\in\mathfrak{D}^\sharp,\ \eta\in\mathfrak{D}^\flat.$
(iii) $F=F^{-1}$.
(iv) *$\Delta:=FS$ is a linear positive non-singular self-adjoint operator such that $\mathfrak{D}(\Delta^{1/2})=\mathfrak{D}^\sharp$.*
(v) *There exists an antilinear isometry J of $\mathfrak{H}$ onto itself such that:*
   a. $(J\xi\mid J\eta)=(\eta\mid\xi),\qquad \xi,\eta\in\mathfrak{H},$

b. $J = J^{-1}$, *equivalently* $J^2 = I$,
c. $J\Delta J = \Delta^{-1}$,
d. $S = J\Delta^{1/2} = \Delta^{-1/2}J$,
e. $F = J\Delta^{-1/2} = \Delta^{1/2}J$.

(vi) J *and* Δ *are uniquely determined by the property* $(v-d)$ *and* $\mathfrak{D}(\Delta^{1/2}) = \mathfrak{D}^\sharp$.

Proof.

(i): From the previous lemma it is clear that $S\mathfrak{D}^\sharp = \mathfrak{D}^\sharp$. Since the involution on $\mathfrak{A}$ is exactly an involution it is injective. Consequently if $S\xi = 0$ for $\xi \in \mathfrak{D}^\sharp$ then letting $\{\xi_n\} \subset \mathfrak{A}$ be the sequence guaranteed by the previous lemma we have that $\lim_n \xi_n^\sharp = 0 \in \mathfrak{A}$. But then $\eta_n := \xi_n^\sharp$ converges in norm to zero and $\{\eta_n^\sharp\} = \{\xi_n\}$ is Cauchy (with limit ξ). Hence $0 = 0^\sharp = \lim_n \xi_n = \xi$. Hence S is injective and S^{-1} exists. Since $S = S^{-1}$ on $\mathfrak{A}$, this holds on all of $\mathfrak{D}^\sharp$.

(ii),(iii): The operator F is merely the adjoint of S. From Proposition X.1.6 in Conway [1] (after making the necessary changes to account for the antilinearity of S) we see that F is densely defined, closed, and with a domain given precisely by $(ii-a)$. The reversal of the vectors in $(ii-b)$ is a consequence of the antilinearity. Also, $F = F^{-1}$ follows from $S = S^{-1}$.

(iv): Define $\Delta = FS$, then Δ is nonsingular as a consequence of S and F being invertible and linear as the composition of two antilinear maps. Note that $\mathfrak{D}(\Delta) = \{\xi \in \mathfrak{D}^\sharp \colon S\xi \in \mathfrak{D}^\flat\}$. For any $\xi \in \mathfrak{D}^\sharp$ we have

$$(\Delta\xi \mid \xi) = (FS\xi \mid \xi) = (S\xi \mid S\xi) = \|S\xi\| \geq 0,$$

ergo Δ is positive. A simple computation shows that $\Delta \subset \Delta^*$, i.e. that Δ is symmetric. Now, suppose $(\xi_n, \Delta\xi_n) \to (\xi_0, \eta_0)$. Then

$$\begin{aligned}\|S\xi_n - S\xi_m\|^2 &= (S(\xi_n - \xi_m) \mid S(\xi_n - \xi_m)) \\ &= (\Delta(\xi_n - \xi_m) \mid \xi_n - \xi_m) \leq \|\Delta\xi_n - \Delta\xi_m\|\|\xi_n - \xi_m\|,\end{aligned}$$

ergo $\{S\xi_n\}$ is a Cauchy sequence. By the previous lemma we then have $\xi_0 \in \mathfrak{D}^\sharp$ and $\lim_n S\xi_n = S\xi_0$. Hence $(S\xi_n, \Delta\xi_n) = (S\xi_n, FS\xi_n) \to (S\xi_0, \eta_0)$ and by the closedness of F we know $\eta_0 = FS\xi_0 = \Delta\xi_0$ and hence Δ is closed. It then follows from Corollary X.2.9 in Conway [1] that Δ is self-adjoint if $\ker(\Delta \pm i) = 0$. But this is true since for example if $\xi \in \ker(\Delta - i) \setminus \{0\}$ then $\Delta\xi = i\xi$ and hence

$$\|\Delta\xi\|^2 = (\Delta\xi \mid \Delta\xi) = -i(\Delta\xi \mid \xi).$$

Since Δ is nonsingular $\|\Delta\xi\|^2 \neq 0$ but $(\Delta\xi \mid \xi) \in \mathbb{R}$ by the positivity of Δ, a contradiction. Similarly $\ker(\Delta + i) = 0$ and so Δ is self-adjoint. Finally, $S = J\Delta^{1/2}$ for some isometry J by the polar decomposition. Then the equality of the domains is immediate.

(v): Let J be as in the polar decomposition of S as above. It is antilinear since S is and hence (a) follows. Since $S = S^{-1}$ we have

$$J\Delta^{-1/2}J^{-1} = JS^{-1} = JS = J^2\Delta^{1/2}.$$

We already know Δ is positive and self-adjoint so $\Delta^{1/2}$ and (after a small computation) $J\Delta^{-1/2}J^{-1}$ is as well. The uniqueness of the polar decomposition and the above equality then implies $J^2 = 1$ or $J = J^{-1}$, which is part (b).

The rest of (d) (the first equality was simply the polar decomposition) then follows: $S = S^{-1} = \Delta^{-1/2}J^{-1} = \Delta^{-1/2}J$. As an invertible isometry we know $J^* = J^{-1} = J$ so that $F = S^* = (J\Delta^{1/2})^* = \Delta^{1/2}J^* = \Delta^{1/2}J$. The other equality in (e) follows from $F = F^{-1}$.

Finally, (c) follows through the use of (d) and (e):

$$J\Delta J = J\Delta^{1/2}\Delta^{1/2}J = SF = \Delta^{-1/2}JJ\Delta^{-1/2} = \Delta^{-1}.$$

(vi): This is merely reiterating the uniqueness of the polar decomposition. □

Definition 1.9. The operators Δ and J from the above lemma are called the **modular operator** and the **modular conjugation** of the left Hilbert algebra $\mathfrak{A}$ respectively.

We will often silently invoke property (ii-b) in the above lemma, so the reader should acquaint themselves with it presently or just remember to refer back to the lemma whenever they do not follow a particular computation.

The involution we need for the construction of a right Hilbert algebra will turn out to be F. Another goal we set for ourselves (it is in fact the *main* goal) is to show that $\mathrm{Ad}(\Delta^{it}) \in \mathrm{Aut}(\mathcal{R}_l(\mathfrak{A}))$ for all $t \in \mathbb{R}$ and that $J\mathcal{R}_l(\mathfrak{A})J = \mathcal{R}_l(\mathfrak{A})'$. We work towards the construction of the right Hilbert algebra $\mathfrak{A}'$ presently (it will turn out to be crucial for establishing the claims regarding the modular operator and conjugation).

Definition 1.10. A vector $\eta \in \mathfrak{H}$ is **right bounded** if

$$\sup\{\|\pi_l(\xi)\eta\| \colon \xi \in \mathcal{A},\ \|\xi\| \leq 1\} < +\infty.$$

The set of right bounded vectors is denoted by $\mathfrak{B}'$.

Clearly $\eta \in \mathfrak{B}'$ iff $\exists a \in \mathcal{B}(\mathfrak{H})$ such that $a\xi = \pi_l(\xi)\eta$ for all $\xi \in \mathfrak{A}$. As this operator is uniquely determined by η, we denote it $\pi_r(\eta) := a$. It is easy to verify that $\mathfrak{B}'$ is a subspace and that π_r is linear.

Lemma 1.11.

(i) $\mathfrak{B}'$ *is invariant under* $\mathcal{R}_l(\mathfrak{A})'$.
(ii) $\mathfrak{n}_r := \pi_r(\mathfrak{B}')$ *is a left ideal of* $\mathcal{R}_l(\mathfrak{A})'$ *and*

$$\pi_r(a\eta) = a\pi_r(\eta), \qquad a \in \mathcal{R}_l(\mathfrak{A})',\ \eta \in \mathfrak{B}'.$$

Proof. Let $a \in \mathcal{R}_l(\mathfrak{A})'$, $\xi \in \mathfrak{A}$ and $\eta \in \mathfrak{B}'$. Then

$$\pi_l(\xi)(a\eta) = a\pi_l(\xi)\eta = a\pi_r(\eta)\xi$$

hence $a\eta \in \mathfrak{B}'$ since $\|a\pi_r(\eta)\xi\| \leq \|a\|\|\pi_r(\eta)\|\|\xi\|$. Moreover, $\pi_r(a\eta) = a\pi_r(\eta)$. If $\zeta \in \mathfrak{A}$, then

$$\pi_r(\eta)\pi_l(\xi)\zeta = \pi_r(\eta)(\xi\zeta) = \pi_l(\xi\zeta)\eta = \pi_l(\xi)\pi_l(\zeta)\eta = \pi_l(\xi)\pi_r(\eta)\zeta,$$

ergo $\pi_r(\eta) \in \mathcal{R}_l(\mathfrak{A})'$. The fact that $\pi_r(\mathcal{B}')$ is an ideal follows from the above work and the comments regarding the linearity of π_r above. $\square$

For the sake of notation we state the following

$$\begin{aligned} \xi\zeta &:= \pi_l(\xi)\zeta, \qquad \xi \in \mathfrak{A},\ \zeta \in \mathfrak{H}; \\ \zeta\eta &:= \pi_r(\eta)\zeta, \qquad \zeta \in \mathfrak{H},\ \eta \in \mathfrak{B}'. \end{aligned}$$

This extended multiplication remains associative due to the commutativity of $\pi_l(\mathfrak{A})$ and $\pi_r(\mathfrak{B}')$:

$$(\xi\zeta)\eta = (\pi_l(\xi)\zeta)\eta = \pi_r(\eta)\pi_l(\xi)\zeta = \pi_l(\xi)\pi_r(\eta)\zeta = \xi(\pi_r(\eta)\zeta) = \xi(\zeta\eta),$$

where $\xi \in \mathfrak{A}$, $\zeta \in \mathfrak{H}$, and $\eta \in \mathfrak{B}'$. We also recall that we may write $\xi^\sharp$ in place of $S\xi$ for $\xi \in \mathfrak{D}^\sharp$, and we define $\eta^\flat := F\eta$ for $\eta \in \mathfrak{D}^\flat$. As the notation suggests, this will be the involution for our right Hilbert algebra. Speaking of which, we lastly define

$$\mathfrak{A}' := \mathfrak{B}' \cap \mathfrak{D}^\flat.$$

Lemma 1.12.

(i) $\pi_r(\mathfrak{B}')^*\mathfrak{B}' \subset \mathfrak{A}'$.
(ii) $(\pi_r(\eta_1)^*\eta_2)^\flat = \pi_r(\eta_2)^*\eta_1, \qquad \eta_1, \eta_2 \in \mathfrak{B}'$.
(iii) $\mathfrak{A}'$ *satisfies (a), (b), and (c) for a right Hilbert algebra.*

Proof. Let $\eta_1, \eta_2 \in \mathfrak{B}'$ and set $\eta = \pi_r(\eta_1)^*\eta_2$. From the previous lemma we know $\mathfrak{B}'$ is $\mathcal{R}_l(\mathfrak{A})'$ invariant and that $\pi_r(\eta_1)^* \in \mathcal{R}_l(\mathfrak{A})'$. Hence $\eta \in \mathfrak{B}'$. Given $\xi \in \mathfrak{A}$ we have

$$\begin{aligned} (\xi^\sharp \mid \eta) &= (\xi^\sharp \mid \pi_r(\eta_1)^*\eta_2) = (\pi_r(\eta_1)\xi^\sharp \mid \eta_2) = (\pi_l(\xi^\sharp)\eta_1 \mid \eta_2) \\ &= (\pi_l(\xi)^*\eta_1 \mid \eta_2) = (\eta_1 \mid \pi_l(\xi)\eta_2) = (\eta_1 \mid \pi_r(\eta_2)\xi) = (\pi_r(\eta_2)^*\eta_1 \mid \xi). \end{aligned}$$

Since this last quantity is bounded by $\|\eta\|\|\xi\|$, we see that $\eta \in \mathfrak{D}^\flat$ and that the formula in (ii) holds.
Since $\mathfrak{A}' \subset \mathfrak{B}'$, (a) is clear. Let $\xi, \zeta \in \mathfrak{A}$ and $\eta \in \mathfrak{A}'$ then

$$(\xi\eta \mid \zeta) = (\eta \mid \xi^\sharp\zeta) = (\zeta^\sharp\xi \mid \eta^\flat) = (\xi \mid \zeta\eta^\flat).$$

The density of $\mathfrak{A}$ in $\mathfrak{H}$ implies this holds for $\xi, \zeta \in \mathfrak{A}'$ as well so (b) holds. We already know the involution is preclosed with closure F. $\square$

Lemma 1.13. *Let $\eta \in \mathfrak{D}^\flat$ and define operators a_0 and b_0 with domain $\mathfrak{A}$ by the following:*

$$a_0\xi := \pi_l(\xi)\eta, \qquad b_0\xi := \pi_l(\xi)\eta^\flat, \qquad \xi \in \mathfrak{A}.$$

Then:

(i) *a_0 and b_0 are preclosed, $a_0 \subset b_0^*$ and $b_0 \subset a_0^*$;*

(ii) *if $\pi_r(\eta) := a_0^{**}$ and $\pi_r(\eta^\flat) := b_0^{**}$, then $\pi_r(\eta)$ and $\pi_r(\eta^\flat)$ are affiliated with $\mathcal{R}_l(\mathfrak{A})'$ in the sense that every unitary in $\mathcal{R}_l(\mathfrak{A})$ commutes with $\pi_r(\eta)$ and $\pi_r(\eta^\flat)$.*

[Note: since we don't necessarily have $\eta \in \mathfrak{B}'$, $\pi_r(\eta)$ may not exist and hence defining a_0 is necessary.]

Proof.

(i): Let $\xi, \zeta \in \mathfrak{A}$, then

$$\begin{aligned}(a_0\xi \mid \zeta) &= (\pi_l(\xi)\eta \mid \zeta) = (\eta \mid \pi_l(\xi)^*\zeta) = (\eta \mid \xi^\sharp\zeta)\\ &= ((\xi^\sharp\zeta)^\sharp \mid \eta^\flat) = (\zeta^\sharp\xi \mid \eta^\flat) = (\xi \mid \pi_l(\zeta)\eta^\flat) = (\xi \mid b_0\zeta),\end{aligned}$$

ergo $a_0 \subset b_0^*$ and (after taking complex conjugates to reverse the inner product) $b_0 \subset a_0^*$. As a_0, b_0 are densely defined, we know a_0^*, b_0^* are closed and thus the previous work implies a_0 and b_0 are preclosed.

(ii): Defining $\pi_r(\eta)$ and $\pi_r(\eta^\flat)$ as above is simply another way of saying that $\pi_r(\eta)$ is the closure of a_0 and $\pi_r(\eta^\flat)$ is the closure of b_0. Given a unitary $u \in \mathcal{R}_l(\mathfrak{A})$, since $ua_0^{**}u^* = (ua_0^*u^*)^*$, it suffices to show that a_0^* is affiliated with $\mathcal{R}_l(\mathfrak{A})'$ (by symmetry it will follow for b_0^{**} as well). We first note that if $\zeta \in \mathfrak{D}(a_0^*)$ and $\xi_1, \xi_2 \in \mathfrak{A}$ then

$$\begin{aligned}[(a_0^*\pi_l(\xi_1)\zeta \mid \xi_2) =](\pi_l(\xi_1)\zeta \mid a_0\xi_2) &= (\pi_l(\xi_1)\zeta \mid \pi_l(\xi_2)\eta) = (\zeta \mid \pi_l(\xi_1^\sharp\xi_2)\eta)\\ &= (\zeta \mid a_0(\xi_1^\sharp\xi_2)) = (a_0^*\zeta \mid \xi_1^\sharp\xi_2) = (\pi_l(\xi_1)a_0^*\zeta \mid \xi_2),\end{aligned}$$

thus $\pi_l(\xi_1)\zeta \in \mathfrak{D}(a_0^*)$ and $a_0^*\pi_l(\xi_1)\zeta = \pi_l(\xi_1)a_0^*\zeta$. Now, let $u \in \mathcal{R}_l(\mathfrak{A})$ be a unitary and let $\{\pi_l(\xi_\alpha)\}$ be a net converging strongly to u. Then by the above calculation we know that for $\zeta \in \mathfrak{D}(a_0^*)$ we have

$$ua_0^*\zeta = \lim_\alpha \pi_l(\xi_\alpha)a_0^*\zeta = \lim_\alpha a_0^*\pi_l(\xi_\alpha)\zeta,$$

so that $u\zeta \in \mathfrak{D}(a_0^*)$ and $ua_0^*\zeta = a_0^*u\zeta$. Hence a_0^* is affiliated with $\mathcal{R}_l(\mathfrak{A})'$. □

Note that since the lemma showed that a_0 and b_0^* agreed on $\mathfrak{A}$, we have $\pi_r(\eta^\flat)\xi = b_0^{**}\xi = (a_0)^*\xi = \pi_r(\eta)^*\xi$ for $\xi \in \mathfrak{A}$.

Lemma 1.14. *Let $\mathcal{K}(0,\infty)$ be the algebra of continuous functions on the open half line $(0,\infty)$ with compact support. For fixed $\eta \in \mathfrak{D}^\flat$, let*

$$\pi_r(\eta) = uh = ku$$

be the left and right polar decompositions. If $f \in \mathcal{K}(0,\infty)$, then $f(h)\eta^\flat$ and $f(k)\eta$ are both right bounded and

$$\begin{aligned}\pi_r(f(h)\eta^\flat) &= hf(h)u^* \in \mathcal{R}_l(\mathfrak{A})',\\ \pi_r(f(k)\eta) &= kf(k)u \in \mathcal{R}_l(\mathfrak{A})'.\end{aligned}$$

Proof. Let $\xi \in \mathfrak{A}$, then since $\pi_r(\eta)$ is affiliated with $\mathcal{R}_l(\mathcal{A})'$ by the previous lemma we know h and $\pi_l(\xi)$ commute, similarly k and $\pi_l(\xi)$ commute. Hence

$$\pi_l(\xi)f(h)\eta^\flat = f(h)\pi_l(\xi)\eta^\flat = f(h)\pi_r(\eta^\flat)\xi = f(h)\pi_r(\eta)^*\xi = f(h)hu^*\xi = hf(h)u^*\xi \qquad \text{and}$$
$$\pi_l(\xi)f(k)\eta = f(k)\pi_l(\xi)\eta = f(k)\pi_r(\eta)\xi = f(k)ku\xi = kf(k)u\xi.$$

Noting that $hf(h)u^*$ and $kf(k)u$ are bounded (u is a partial isometry and $hf(h)$ and $kf(k)$ are bounded by the functional calculus), we see that $f(h)\eta^\flat$ and $f(k)\eta$ are indeed right bounded and that desired formulas hold. □

Lemma 1.15. *$\mathfrak{A}'$ and $(\mathfrak{A}')^2$ are both dense in $\mathfrak{D}^\flat$ with respect to the norm $\|\cdot\|_\flat$ defined by*

$$\|\eta\|_\flat := \sqrt{\|\eta\|^2 + \|\eta^\flat\|^2}, \qquad \eta \in \mathfrak{D}^\flat.$$

In particular, they are both dense in $\mathfrak{H}$, whence $\mathfrak{A}'$ is a right Hilbert algebra.

Proof. The density of $\mathfrak{A}'$ and $(\mathfrak{A}')^2$ in $\mathfrak{H}$ will follow from their density in $\mathfrak{D}^\flat$ under the norm $\|\cdot\|_\flat$ since this norm dominates the standard norm and since $\mathfrak{D}^\flat$ is itself dense in $\mathfrak{H}$. Also, we already know $\mathfrak{A}'$ satisfies properties (a), (b), and (c) for a right Hilbert algebra. So the fact that $(\mathfrak{A}')^2$ is dense in $\mathfrak{H}$ implies it is dense in $\mathfrak{A}'$ and we get property (d). (It was never mentioned before, but the containment $(\mathfrak{A}')^2 \subset \mathfrak{A}'$ is clear since $\mathfrak{B}'$ and $\mathfrak{D}^\flat$ are closed under multiplication.)

Recall

$$\mathfrak{n}_r = \pi_r(\mathfrak{B}').$$

Let $\eta \in \mathfrak{D}^\flat$ and recall the notation from the previous lemma. Since $uh = ku$ we also have $hu^* = u^*k$, hence we can move u right across h to convert it into a k and u^* right across k to convert it into a h. So for each $f \in \mathcal{K}(0,\infty)$ we have

$$\begin{aligned} hf(h) &= u^*uhf(h) = u^*kf(k)u = u^*\pi_r(f(k)\eta) = \pi_r(u^*f(k)\eta) \in \mathfrak{n}_r, \\ kf(k) &= uu^*kf(k) = uhf(h)u^* = u\pi_r(f(h)\eta) = \pi_r(uf(h)\eta) \in \mathfrak{n}_r. \end{aligned}$$

Let $g \in \mathcal{K}(0,\infty)$ be such that $f(\lambda) = \lambda g(\lambda)$, $\lambda > 0$. Then $f(h) = hg(h) \in \mathfrak{n}_r$ and $f(k) = kg(k) \in \mathfrak{n}_r$ for all $f \in \mathcal{K}(0,\infty)$. Also, letting $f_1, f_2 \in \mathcal{K}(0,\infty)$ be such that $f(\lambda) = \overline{f_1(\lambda)}f_2(\lambda)$ for $\lambda > 0$ we have

$$f(h) = f_1(h)^*f_2(h) \in \mathfrak{n}_r^*\mathfrak{n}_r \qquad \text{and} \qquad f(k) = f_1(k)^*f_2(k) \in \mathfrak{n}_r^*\mathfrak{n}_r.$$

From Lemma 1.12(i) we know that $\mathfrak{n}_r^*\mathfrak{n}_r \subset \pi_r(\mathfrak{A}')$, so $f(h), f(k) \in \pi_r(\mathfrak{A}')$ for all $f \in \mathcal{K}(0,\infty)$. Hence $f(h) = \pi_r(\eta_1)$ and $f(k) = \pi_r(\eta_2)$ and so $f(h)\eta = \eta\eta_1$ and $f(k)\eta^\flat = \eta^\flat\eta_2$. Since $\mathfrak{D}^\flat$ is closed under multiplication, we have $f(h)\eta, f(k)\eta^\flat \in \mathfrak{D}^\flat$ and hence are in $\mathfrak{A}'$ since the previous lemma showed these elements were right bounded. In fact we have even showed $f(h)\eta, f(k)\eta^\flat \in (\mathfrak{A}')^2$. Note that the formulas from the previous lemma yield

$$\pi_r((f(k)\eta)^\flat) = \pi_r(f(k)\eta)^* = (kf(k)u)^* = u^*kf(k)^* = hf(h)^*u^* = \pi_r(f(h)^*\eta^\flat),$$

or simply

$$(f(k)\eta)^\flat = f(h)^*\eta^\flat. \tag{1}$$

Let $\{f_n\}$ be a positive increasing sequence in $\mathcal{K}(0,\infty)$ converging to 1 for $\lambda > 0$. Then $\{f_n(h)\}$ and $\{f_n(k)\}$ converge strongly to the range projections p and q of h and k respectively. Since $\pi_r(\eta)^* = (uh)^* = hu^*$, its range projection is the same as p and the range projection of $\pi_r(\eta)$ is the same as q. Suppose for the moment that $q\eta = \eta$ and $p\eta^\flat = \eta^\flat$. Then $\{f_n(k)\eta\}$ converges to η and $\{f_n(h)\eta^\flat\}$ to $\eta^\flat$. Since $(f_n(k)\eta)^\flat = f_n(k)^*\eta^\flat = f_n(k)\eta^\flat$, we will have that $\{f_n(k)\eta\}$ converges to η with respect to $\|\cdot\|_\flat$ and hence $(\mathfrak{A}')^2$ will be dense in $\mathfrak{D}^\flat$. To see that $q\eta = \eta$ and $p\eta^\flat = \eta^\flat$, we'll show $\eta \in q\mathfrak{H}$ and $\eta^\flat \in p\mathfrak{H}$. Let $\{\pi_l(\xi_\alpha)\} \subset \pi_l(\mathfrak{A})$ be a net converging strongly to the identity in $\mathcal{R}_l(\mathfrak{A})$. Then we have

$$\begin{aligned} \eta &= \lim_\alpha \pi_l(\xi_\alpha)\eta = \lim_\alpha \pi_r(\eta)\xi_\alpha \in q\mathfrak{H}; \\ \eta^\flat &= \lim_\alpha \pi_l(\xi_\alpha)\eta^\flat = \lim_\alpha \pi_r(\eta)^*\xi_\alpha \in p\mathfrak{H}. \end{aligned}$$

□

Theorem 1.16. $\mathcal{R}_l(\mathfrak{A})' = \mathcal{R}_r(\mathfrak{A}')$.

Proof. We have seen $\pi_r(\mathfrak{A}') \subset \mathcal{R}_l(\mathfrak{A})'$, so $\mathcal{R}_r(\mathfrak{A}') \subset \mathcal{R}_l(\mathfrak{A})'$.

Conversely, the density of $(\mathfrak{A}')^2$ asserts that $\pi_r(\mathfrak{A}')$ is a non-degenerate $*$-subalgebra of $\mathcal{R}_l(\mathfrak{A})'$. So the identity in $\mathcal{R}_r(\mathfrak{A}')$ is the identity in $\mathcal{B}(\mathfrak{H})$, and hence we can find a bounded net $\{a_i\} \subset \pi_r(\mathfrak{A}')$ converging σ-strongly* to 1. Then for $x \in \mathcal{R}_l(\mathfrak{A})'$ we have $x = \lim a_i^*xa_i$, but $a_i^*xa_i \in \mathfrak{n}_r^*\mathfrak{n}_r \subset \pi_r(\mathfrak{A}')$. Hence $\mathcal{R}_l(\mathfrak{A})' = \mathcal{R}_r(\mathfrak{A}')$. □

Lemma 1.17.

(i) $\mathfrak{A}^2$ *is dense in* $\{\mathfrak{D}^\sharp, \|\cdot\|_\sharp\}$. *In particular this implies that if* $\eta_1, \eta_2 \in \mathfrak{H}$ *satisfy*

$$(\xi_1^\sharp\xi_2 \mid \eta_1) = (\eta_2 \mid \xi_2^\sharp\xi_1), \qquad \xi_1, \xi_2 \in \mathfrak{A},$$

then $\eta_1 \in \mathfrak{D}^\flat$ *and* $\eta_1^\flat = \eta_2$.

(ii) $\pi_r(\mathfrak{A}') = \mathfrak{n}_r \cap \mathfrak{n}_r^*$.

Proof.

(i): Fix $\xi \in \mathfrak{A}$. Let $\{\pi_r(\eta_\alpha)\} \subset \pi_r(\mathfrak{A}')$ be a net converging strongly to 1, then $\xi = \lim_\alpha \pi_r(\eta_\alpha)\xi = \lim_\alpha \pi_l(\xi)\eta_\alpha$ and hence $\xi \in [\pi_l(\xi)\mathfrak{H}]$. Scaling if necessary, we can assume $\|\pi_l(\xi)\| \leq 1$. Define a function $p_n \colon \mathbb{R} \to \mathbb{R}$ by

$$p_n(t) := 1 - (1-t)^n.$$

By the functional calculus, for any operator a with $\|a\| \leq 1$, $p_n(aa^*)$ converges to the range projection $s_l(a)$. By the above work we know $s_l(\pi_l(\xi))\xi = \xi$, similarly we know $s_l(\pi_l(\xi^\sharp))\xi^\sharp = \xi^\sharp$. So we compute

$$\begin{aligned} \xi = s_l(\xi)\xi &= \lim_{n\to\infty} p_n(\pi_l(\xi)\pi_l(\xi)^*)\xi = \lim_{n\to\infty} p_n(\xi\xi^\sharp)\xi; \\ \xi^\sharp = s_l(\pi_l(\xi^\sharp))\xi^\sharp &= \lim_{n\to\infty} p_n(\pi_l(\xi^\sharp)\pi_l(\xi^\sharp)^*)\xi^\sharp \\ &= \lim_{n\to\infty} p_n(\xi^\sharp\xi)\xi^\sharp = \lim_{n\to\infty} \xi^\sharp p_n(\xi\xi^\sharp) = \lim_{n\to\infty} (p_n(\xi\xi^\sharp)\xi)^\sharp. \end{aligned}$$

(To convince yourself of the second to last equality, check it for p_n a monomial.) Hence ξ is the $\|\cdot\|_\sharp$-limit of elements in $\mathfrak{A}^2$.

If η_1, η_2 satisfy the above relation it is the same as

$$(S(\xi_2^\sharp\xi_1) \mid \eta_1) = (\eta_1 \mid \xi_2^\sharp\xi_1),$$

hence if $\|\xi_2^\sharp\xi_1\| \leq 1$ then

$$|(S(\xi_2^\sharp\xi_1) \mid \eta_1)| \leq \|\eta_2\|.$$

Thus the density of $\mathfrak{A}^2$ in the domain of S, which we just established above, shows that $\eta_1 \in \mathfrak{D}^\flat$ and $F\eta_1 = \eta_1^\flat = \eta_2$.

(ii): The containment $\pi_r(\mathfrak{A}') \subset \mathfrak{n}_r$ follows from the definition of $\mathfrak{A}'$. To see that $\pi_r(\mathfrak{A}') \subset \mathfrak{n}_r^*$, simply note that for $\eta \in \mathfrak{D}^\flat$ we know $\pi_r(\eta) = \pi_r(\eta)^{**} = \pi_r(\eta^\flat)^*$.

Conversely, suppose $\pi_r(\eta_1) \in \mathfrak{n}_r \cap \mathfrak{n}_r^*$. Then $\exists \eta_2 \in \mathfrak{B}'$ such that $\pi_r(\eta_1)^* = \pi_r(\eta_2)$. For $\xi_1, \xi_2 \in \mathfrak{A}$ we then have

$$\begin{aligned} (\xi_1^\sharp\xi_2 \mid \eta_1) &= (\xi_2 \mid \xi_1\eta_1) = (\xi_2 \mid \pi_r(\eta_1)\xi_1) = (\pi_r(\eta_1)^*\xi_2 \mid \xi) \\ &= (\pi_r(\eta_2)\xi_2 \mid \xi_1) = (\xi_2\eta_2 \mid \xi_1) = (\eta_2 \mid \xi_2^\sharp\xi_1), \end{aligned}$$

so that by part (i) $\eta_1 \in \mathfrak{D}^\flat$ (hence $\eta_1 \in \mathfrak{A}'$) and $\eta_1^\flat = \eta_2$. Thus $\mathfrak{n}_r \cap \mathfrak{n}_r^* \subset \pi_r(\mathfrak{A}')$. □

Starting from the right Hilbert algebra $\mathfrak{A}'$ we can dualize the above treatment of left Hilbert algebras. The dual results will have the same numbering as their counterparts but will be marked with a prime: $'$.

Definition 1.10$'$. A vector $\xi \in \mathfrak{H}$ is **left bounded** if

$$\sup\{\|\pi_r(\eta)\xi\| \colon \eta \in \mathfrak{A}',\ \|\eta\| \leq 1\} < +\infty.$$

The set of left bounded vectors is denoted $\mathfrak{B}$.

We know $\mathfrak{A} \subset \mathfrak{B}$ and we can associate a bounded operator $\pi_l(\xi)$ on $\mathfrak{H}$ to each $\xi \in \mathfrak{B}$ in the obvious way.

Lemma 1.11$'$.

(i) *$\mathfrak{B}$ is invariant under $\mathcal{R}_l(\mathfrak{A})$.*

(ii) *$\mathfrak{n}_l := \pi_l(\mathfrak{B})$ is a left ideal of $\mathcal{R}_l(\mathfrak{A})$ and*

$$\pi_l(a\xi) = a\pi_l(\xi), \qquad a \in \mathcal{R}_l(\mathfrak{A}),\ \xi \in \mathfrak{B}.$$

The proof is similar enough to the original Lemma 1.11 that we leave it to the reader.

We once again extend multiplication to this new class of vectors so that $\xi\eta := \pi_l(\xi)\eta$ whenever $\xi \in \mathfrak{B}$ and $\eta \in \mathfrak{H}$. Note that this is consistent with our previous extension as well. Define

$$\mathfrak{A}'' = \mathfrak{B} \cap \mathfrak{D}^\sharp.$$

Then dual of the arguments which showed $\mathfrak{A}'$ is a right Hilbert algebra give us that $\mathfrak{A}''$ is a left Hilbert algebra such that $\mathfrak{A} \subset \mathfrak{A}''$ and $\mathcal{R}_l(\mathfrak{A}'') = \mathcal{R}_r(\mathfrak{A}')'$. But the later von Neumann algebra is nothing more than the left von Neumann algebra of $\mathfrak{A}$, i.e. $\mathcal{R}_l(\mathfrak{A}'') = \mathcal{R}_l(\mathfrak{A})$.

Lemma 1.17$'$.

(ii$'$) $\pi_l(\mathfrak{A}'') = \mathfrak{n}_l \cap \mathfrak{n}_l^*$.

Again, this follows by a similar argument to that of Lemma 1.17.

Iterating this dualization we would obtain

$$\mathfrak{A} \subset \mathfrak{A}'' = \mathfrak{A}^{(iv)} = \cdots ,$$
$$\mathfrak{A}' = \mathfrak{A}''' = \mathfrak{A}^{(v)} = \cdots .$$

Definition 1.18. We say a left Hilbert algebra $\mathfrak{A}$ is **full** if $\mathfrak{A} = \mathfrak{A}''$. Given two left Hilbert algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, we say they are **equivalent** if $\mathfrak{A}_1''$ and $\mathfrak{A}_2''$ are isometrically $*$-isomorphic.

So for example, $\mathfrak{A}$ and $\mathfrak{A}''$ are equivalent. As it does not affect the relevant von Neumann algebras, we henceforth assume $\mathfrak{A}$ is full.

Lemma 1.13′. *Let $\xi \in \mathfrak{D}^\sharp$ and define operators a_0 and b_0 with domain $\mathfrak{A}'$ by the following:*

$$a_0\eta := \pi_r(\eta)\xi, \qquad b_0\eta := \pi_r(\eta)\xi^\sharp, \qquad \eta \in \mathfrak{A}'.$$

Then:

(i) *a_0 and b_0 are preclosed, $a_0 \subset b_0^*$ and $b_0 \subset a_0^*$;*
(ii) *if $\pi_l(\xi) := a_0^{**}$ and $\pi_l(\xi^\sharp) := b_0^{**}$, then $\pi_l(\xi)$ and $\pi_l(\xi^\sharp$ are affiliated with $\mathcal{R}_l(\mathfrak{A})$.*

Between us and the main objective is an onslaught of technical lemmas. Readers who are faint of heart may wish to simply skip to Theorem 1.24, others are encouraged to get a cup of coffee.

Lemma 1.19. *For each $\omega \in \mathbb{C} \setminus \mathbb{R}_+$ set*

$$\gamma(\omega) := \frac{1}{\sqrt{2(|\omega| - Re\ \omega)}}.$$

(i) *$(\Delta - \omega)^{-1}\mathfrak{A}' \subset \mathfrak{A}$ and*
$$\left\|\pi_l((\Delta - \omega)^{-1})\eta)\right\| \leq \gamma(\omega)\|\pi_r(\eta)\|, \qquad \eta \in \mathfrak{A}'.$$
(ii) *$(\Delta^{-1} - \omega)^{-1}\mathfrak{A} \subset \mathfrak{A}'$ and*
$$\left\|\pi_r((\Delta^{-1} - \omega)^{-1}\xi)\right\| \leq \gamma(\omega)\|\pi_l(\xi)\|, \qquad \xi \in \mathfrak{A}.$$

Proof. We only prove (i) as the symmetry of the argument will allow (ii) to follow easily. Fix $\eta \in \mathfrak{A}'$ and set $\xi := (\Delta - \omega)^{-1}\eta$. We know that $\xi \in \mathfrak{D}(\Delta) \subset \mathfrak{D}^\sharp$ with $\Delta\xi = \omega^{-1}(\omega^{-1} - \Delta^{-1})^{-1}\eta$. Let $\pi_l(\xi)$ be as in Lemma 1.13′(ii), and let $\pi_l(\xi) = uh = ku$ be the left and right polar decompositions. By the dual of Lemma 1.14 we know that $f(k)\xi \in \mathfrak{A}$ for every $f \in \mathcal{K}(0,\infty)$ and that

$$(f(k)\xi)^\sharp = f(h)^*\xi^\sharp, \qquad f \in \mathcal{K}(0,\infty).$$

We also have

$$\begin{aligned}
2(|w| - \text{Re }\omega)\|hf(h)\xi^\sharp\|^2 &= 2(|\omega| - \text{Re }\omega)(hf(h)^*hf(h)\xi^\sharp \mid \xi^\sharp) = 2(|\omega| - \text{Re }\omega)(\{kf(k)^*kf(k)\xi\}^\sharp \mid \xi^\sharp)\\
&= 2(|\omega| - \text{Re }\omega)(\Delta\xi \mid kf(k)^*kf(k)\xi)\\
&\leq 2|\omega|\|kf(k)\Delta\xi\|\|kf(k)\xi\| - 2\text{Re }\omega(kf(k)\Delta\xi \mid kf(k)\xi)
\end{aligned}$$

Recall algebra: $(\|kf(k)\Delta\xi\| - |\omega|\|kf(k)\xi\|)^2 \geq 0$ so that $2|\omega|\|kf(k)\Delta\xi\|\|kf(k)\xi\| \leq \|kf(k)\Delta\xi\|^2 + |\omega|^2\|kf(k)\xi\|^2$. Continuing the above computation with this we have:

$$\begin{aligned}
2(|w| - \text{Re }\omega)\|hf(h)\xi^\sharp\|^2 &\leq \|kf(k)\Delta\xi\|^2 + |\omega|^2\|kf(k)\xi\|^2 - 2\text{Re }\omega(kf(k)\Delta\xi \mid kf(k)\xi)\\
&= \|kf(k)(\Delta - \omega)\xi\|^2 = \|kf(k)\eta\|^2 = \|f(k)k\eta\|^2 = \|f(k)uu^*k\eta\|^2\\
&= \|f(k)u\pi_l(\xi)^*\eta\|^2 = \|f(k)u\pi_l(\xi^\sharp)\eta\|^2 = \|f(k)u\pi_r(\eta)\xi^\sharp\|^2\\
&= \|\pi_r(\eta)f(k)u\xi^\sharp\|^2 = \|\pi_r(\eta)uf(h)\xi^\sharp\|^2 \leq \|\pi_r(\eta)\|^2\|f(h)\xi^\sharp\|^2.
\end{aligned}$$

Thus we have established

$$\|hf(h)\xi^\sharp\| \leq \gamma(\omega)\|\pi_r(\eta)\|\|f(h)\xi^\sharp\|$$

Let

$$h = \int_0^\infty \lambda\ dE(\lambda)$$

be the spectral decomposition of h. Then the above inequality implies

$$\int_0^\infty \lambda |f(\lambda)|\ d\|E(\lambda)\xi^\sharp\| \leq c\int_0^\infty |f(\lambda)|\ d\|E(\lambda)\xi^\sharp\|, \qquad f \in \mathcal{K}(0,\infty),$$

where $c := \gamma(\omega)\|\pi_r(\eta)\|$. But this implies that the spectral measure $d\|E(\lambda)\xi^\sharp\|$ is supported on $[0,c]$. Hence $E([0,c])\xi^\sharp = \xi^\sharp$. Recall that since h is affiliated with $\mathcal{R}_l(\mathfrak{A})$, E commutes with $\mathcal{R}_r(\mathfrak{A}') = \mathcal{R}_l(\mathfrak{A})'$. So for any $\zeta \in \mathfrak{A}'$ we have

$$E([0,c])\pi_l(\xi)^*\zeta = E([0,c])\pi_l(\xi^\sharp)\zeta = E([0,c])\pi_r(\zeta)\xi^\sharp = \pi_r(\zeta)E([0,c])\xi^\sharp = \pi_r(\zeta)\xi^\sharp = \pi_l(\xi^\sharp)\zeta,$$

thus

$$\|\pi_l(\xi^\sharp)\zeta\| = \|E([0,c])\pi_l(\xi)^*\zeta\| = \|E([0,c])hu^*\zeta\| \leq c\|u^*\zeta\| = c\|\zeta\|.$$

So $\xi^\sharp$ is left bounded with $\|\pi_l(\xi^\sharp)\| \leq c$. But then ξ is left bounded with

$$\|\pi_l(\xi)\| = \|\pi_l(\xi)^*\| = \|\pi_l(\xi^\sharp)\| \leq c = \gamma(\omega)\|\pi_r(\eta)\|.$$

□

Lemma 1.20. *For $\eta \in \mathfrak{A}'$, set $\xi = (\Delta + s)^{-1}\eta$ for $s > 0$. Then for each $\zeta_1, \zeta_2 \in \mathfrak{D}(\Delta^{1/2}) \cap \mathfrak{D}(\Delta^{-1/2})$ we have*

$$(\pi_r(\eta)\zeta_1 \mid \zeta_2) = (J\pi_l(\xi)^*J\Delta^{-1/2}\zeta_1 \mid \Delta^{1/2}\zeta_2) + s(J\pi_l(\xi)^*J\Delta^{1/2}\zeta_1 \mid \Delta^{-1/2}\zeta_2).$$

Proof. First suppose $\zeta_1, \zeta_2 \in \mathfrak{A} \cap \mathfrak{D}(\Delta^{-1/2})$ (i.e. we take as an extra hypothesis that ζ_1, ζ_2 are left bounded). We compute

$$\begin{aligned}
(\pi_r(\eta)\zeta_1 \mid \zeta_2) &= (\pi_l(\zeta_1)\eta \mid \zeta_2) = (\eta \mid \zeta_1^\sharp\zeta_2) = ((\Delta+s)\xi \mid \zeta_1^\sharp\zeta_2) = (FS\xi \mid \zeta_1^\sharp\zeta_2) + s(\xi \mid \zeta_1^\sharp\zeta_2)\\
&= (\zeta_2^\sharp\zeta_1 \mid S\xi) + s(\zeta_1\xi \mid \zeta_2) = (\zeta_1 \mid \zeta_2\xi^\sharp) + s((\xi^\sharp\zeta_1^\sharp)^\sharp \mid \zeta_2) = (\zeta_1 \mid (\xi\zeta_2^\sharp)^\sharp) + s((\xi^\sharp\zeta_1^\sharp)^\sharp \mid \zeta_2)\\
&= (\zeta_1 \mid \Delta^{-\frac{1}{2}}J\pi_l(\xi)J\Delta^{\frac{1}{2}}\zeta_2) + s(\Delta^{-\frac{1}{2}}J\pi_l(\xi)^*J\Delta^{\frac{1}{2}}\zeta_1 \mid \zeta_2)\\
&= (\Delta^{-\frac{1}{2}}\zeta_1 \mid J\pi_l(\xi)J\Delta^{\frac{1}{2}}\zeta_2) + s(J\pi_l(\xi)^*J\Delta^{\frac{1}{2}}\zeta_1 \mid \Delta^{-\frac{1}{2}}\zeta_2)\\
&= (J\pi_l(\xi)^*J\Delta^{-\frac{1}{2}}\zeta_1 \mid \Delta^{\frac{1}{2}}\zeta_2) + s(J\pi_l(\xi)^*J\Delta^{\frac{1}{2}}\zeta_1 \mid \Delta^{-\frac{1}{2}}\zeta_2).
\end{aligned}$$

Thus the formula holds in this case. Noting that both sides are sesquilinear forms (we are using the fact that $\xi \in \mathfrak{A}$ by the previous lemma with $\omega = -s$), it suffices to show that we can approximate $\zeta \in \mathfrak{D}(\Delta^{1/2}) \cap \mathfrak{D}(\Delta^{-1/2})$ by a sequence $\{\zeta_n\} \subset \mathfrak{A} \cap \mathfrak{D}(\Delta^{-1/2})$ in the sense that

$$\lim_{n\to\infty}\|\zeta - \zeta_n\| = 0, \qquad \lim_{n\to\infty}\|\Delta^{\frac{1}{2}}(\zeta-\zeta_n)\| = 0, \qquad \lim_{n\to\infty}\|\Delta^{-\frac{1}{2}}(\zeta-\zeta_n)\| = 0.$$

Now, $\Delta^{-1/2}\mathfrak{A}' = JF\mathfrak{A}' = J\mathfrak{A}'$ is dense in $\mathfrak{H}$ since $\mathfrak{A}'$ is and J is an antilinear isometry. So for $\zeta \in \mathfrak{D}(\Delta^{1/2}) \cap \mathfrak{D}(\Delta^{-1/2})$ there is a sequence $\{\eta_n\} \subset \mathfrak{A}'$ such that

$$(\Delta^{\frac{1}{2}} + \Delta^{-\frac{1}{2}})\zeta = \lim_{n\to\infty}\Delta^{-\frac{1}{2}}\eta_n.$$

Set $\zeta_n := (1+\Delta)^{-1}\eta_n$. Then by the previous lemma we know $\zeta_n \in \mathfrak{A} \cap \mathfrak{D}(\Delta^{-1/2})$ and since $\Delta^{-1/2}(\Delta^{1/2} + \Delta^{-1/2})^{-1} = (1+\Delta)^{-1}$ we have

$$\zeta = (\Delta^{\frac{1}{2}} + \Delta^{-\frac{1}{2}})^{-1}\lim_{n\to\infty}\Delta^{-\frac{1}{2}}\eta_n = \lim_{n\to\infty}(1+\Delta)^{-1}\eta_n = \lim_{n\to\infty}\zeta_n.$$

We are able to pass $(\Delta^{1/2} + \Delta^{-1/2})^{-1}$ through the limit as a consequence of $\zeta_n \in \mathfrak{A} \cap \mathfrak{D}(\Delta^{-1/2}) \subset \mathfrak{D}(\Delta^{1/2}) \cap \mathfrak{D}(\Delta^{-1/2})$. Also

$$\begin{aligned}
\Delta^{\frac{1}{2}}\zeta &= \Delta^{\frac{1}{2}}(\Delta^{\frac{1}{2}} + \Delta^{-\frac{1}{2}})^{-1}(\Delta^{\frac{1}{2}} + \Delta^{-\frac{1}{2}})\zeta = \Delta(1+\Delta)^{-1}\lim_{n\to\infty}\Delta^{-\frac{1}{2}}\eta_n\\
&= \lim_{n\to\infty}\Delta^{\frac{1}{2}}(1+\Delta)^{-1}\eta_n = \lim_{n\to\infty}\Delta^{\frac{1}{2}}\zeta_n;\\
\Delta^{-\frac{1}{2}}\zeta &= \Delta^{-\frac{1}{2}}(\Delta^{\frac{1}{2}} + \Delta^{-\frac{1}{2}})^{-1}(\Delta^{\frac{1}{2}} + \Delta^{-\frac{1}{2}})\zeta = (1+\Delta)^{-1}\lim_{n\to\infty}\Delta^{-\frac{1}{2}}\eta_n\\
&= \lim_{n\to\infty}\Delta^{-\frac{1}{2}}(1+\Delta)^{-1}\eta_n = \lim_{n\to\infty}\Delta^{-\frac{1}{2}}\zeta_n.
\end{aligned}$$

□

Lemma 1.21. *Let A be a unital Banach algebra. Suppose $\{u(\alpha)\colon \alpha \in \mathbb{C}\}$ is a complex one parameter subgroup of $GL(A)$ of invertible elements in A, i.e.*

$$u(\alpha+\beta) = u(\alpha)u(\beta), \qquad \alpha,\beta\in\mathbb{C}.$$

Furthermore, assume that $\alpha \mapsto u(\alpha)$ is holomorphic and

$$\sup\{\|u(t)\|\colon t\in\mathbb{R}\} = M < +\infty.$$

Then for any $s\in\mathbb{R}$, $e^{-s/2}u(-i/2)+e^{s/2}u(i/2)$ is invertible and

$$\left[e^{-\frac{s}{2}}u\left(-\frac{i}{2}\right) + e^{\frac{s}{2}}u\left(\frac{i}{2}\right)\right]^{-1} = \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}} u(t)\ dt.$$

Proof. Fix $s\in\mathbb{R}$ and set

$$f(\alpha) := \frac{e^{-is\alpha}}{e^{\pi\alpha}-e^{-\pi\alpha}}u(\alpha), \qquad \alpha\in\mathbb{C}.$$

Then f is a meromorphic A-valued function with simple poles at $\alpha = in$, $n\in\mathbb{Z}$. Let $\alpha = r+it$ for $r,t\in\mathbb{R}$. Then $e^{-is\alpha} = e^{st}e^{-isr}$ and $u(\alpha) = u(r)u(it)$. So our hypothesis give us the bound

$$\|f(\alpha)\| \leq Me^{st}\frac{1}{|e^{\pi\alpha}-e^{-\pi\alpha}|}\|u(it)\|.$$

For $R>0$ we define C_R to be rectangular curve in $\mathbb{C}$ with vertices $\pm R\pm \frac{i}{2}$, oriented counter clock-wise. Note that for all R, C_R only encloses one pole of $f(\alpha)$, namely $\alpha=0$. The above bound implies that for fixed $t=\frac{1}{2}$, as $|r|\to\infty$ we have $|r|\|f(\alpha)\|\to 0$. Hence we can compute the following limit as

$$I := \lim_{R\to\infty}\oint_{C_R} f(\alpha)\ d\alpha = \int_{-\infty}^{\infty} f\left(r-\frac{i}{2}\right)\ dr - \int_{-\infty}^{\infty} f\left(r+\frac{i}{2}\right)\ dr.$$

On the other hand, the residue theorem yields

$$I = 2\pi i\lim_{\alpha\to 0}\alpha f(\alpha) = i.$$

Recall complex analysis: $e^{\pi(r\pm i/2)} = e^{\pi r} i\sin(\pm\pi/2) = \pm ie^{\pi r}$ and similarly $-e^{-\pi(r\pm i/2)} = \pm ie^{-\pi r}$. Hence we have

$$\begin{aligned}
i = I &= \int_{-\infty}^{\infty}\frac{e^{-is(r-\frac{i}{2})}}{e^{\pi(r-\frac{i}{2})}-e^{-\pi(r-\frac{i}{2})}}u\left(r-\frac{i}{2}\right)\ dr - \int_{-\infty}^{\infty}\frac{e^{-is(r+\frac{i}{2})}}{e^{\pi(r+\frac{i}{2})}-e^{-\pi(r+\frac{i}{2})}}u\left(r+\frac{i}{2}\right)\ dr\\
&= \frac{e^{-\frac{s}{2}}}{-i}u\left(-\frac{i}{2}\right)\int_{-\infty}^{\infty}\frac{e^{-isr}}{e^{\pi r}+e^{-\pi r}}u(r)\ dr - \frac{e^{\frac{s}{2}}}{i}u\left(\frac{i}{2}\right)\int_{-\infty}^{\infty}\frac{e^{-isr}}{e^{\pi r}+e^{-\pi r}}u(r)\ dr\\
&= i\left[e^{-\frac{s}{2}}u\left(-\frac{i}{2}\right)+e^{\frac{s}{2}}u\left(\frac{i}{2}\right)\right]\int_{-\infty}^{\infty}\frac{e^{-isr}}{e^{\pi r}+e^{-\pi r}}u(r)\ dr,
\end{aligned}$$

which yields the desired equality upon replacing r with t. □

We will invoke the equality established in the preceding lemma in the following equivalent form:

$$e^{\frac{s}{2}}u\left(-\frac{i}{2}\right)(u(-i)+e^s)^{-1} = \int_{-\infty}^{\infty}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}u(t)\ dt, \qquad s\in\mathbb{R}.$$

The relevant one parameter unitary group we will apply this to is Δ^{it}; however, the fact that Δ is unbounded requires some work before we can directly apply it:

Lemma 1.22. *If Δ is the modular operator for a full left Hilbert algebra $\mathfrak{A}$ then we have*

$$e^{\frac{s}{2}}\Delta^{\frac{1}{2}}(\Delta+e^s)^{-1} = \int_{-\infty}^{\infty}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\Delta^{it}\ dt, \qquad s\in\mathbb{R}.$$

Proof. Let

$$\Delta = \int_0^\infty \lambda dE(\lambda)$$

be the spectral decomposition and set $E_r := E([1/r, r])$ for $r > 1$. We apply the preceding lemma to the unital Banach algebra $A = \mathcal{B}(E_r\mathfrak{H})$ and $u(\alpha) = (\Delta E_r)^{i\alpha}$, $\alpha \in \mathbb{C}$ to obtain

$$e^{\frac{s}{2}}\Delta^{\frac{1}{2}}(\Delta + e^s)^{-1}E_r = \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}}\Delta^{it}E_r\ dt, \qquad s \in \mathbb{R}.$$

Letting $r \to \infty$ we conclude the lemma. □

We prove one final technical lemma before we arrive at the much anticipated result.

Lemma 1.23. *Let Δ be the modular operator for a full left Hilbert algebra $\mathfrak{A}$. If $x, y \in \mathcal{B}(\mathfrak{H})$ and $s \in \mathbb{R}$ satisfy the following equation for every $\zeta_1, \zeta_2 \in \mathfrak{D}(\Delta^{1/2}) \cap \mathfrak{D}(\Delta^{-1/2})$:*

$$(x\zeta_1 \mid \zeta_2) = (y\Delta^{-\frac{1}{2}}\zeta_1 \mid \Delta^{\frac{1}{2}}\zeta_2) + e^s(y\Delta^{\frac{1}{2}}\zeta_1 \mid \Delta^{-\frac{1}{2}}\zeta_2),$$

then

$$e^{\frac{s}{2}}y = \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}}\Delta^{it}x\Delta^{-it}\ dt.$$

Proof. For $r > 1$ let E_r be as in the previous proof, and set $A = \mathcal{B}(\mathcal{B}(E_r\mathfrak{H}))$ and define $u(\alpha) = \sigma_\alpha$ where

$$\sigma_\alpha(x) = \Delta^{i\alpha}x\Delta^{-i\alpha}, \qquad x \in \mathcal{B}(E_r\mathfrak{H}).$$

Note that $\|u(t)\| = 1$ for each $t \in \mathbb{R}$ since Δ^{it} is unitary and that the map $\alpha \mapsto u(\alpha)$ is holomorphic. By assumption we have, for each $\zeta_1, \zeta_2 \in \mathfrak{H}$,

$$\begin{aligned}(E_r x E_r\zeta_1 \mid \zeta_2) &= (\Delta^{\frac{1}{2}}E_r y\Delta^{-\frac{1}{2}}E_r\zeta_1 \mid \zeta_2) + e^s(\Delta^{-\frac{1}{2}}E_r y\Delta^{\frac{1}{2}}E_r\zeta_1 \mid \zeta_2)\\ &= \left(\left[\sigma_{-\frac{i}{2}}(E_r y E_r) + e^s\sigma_{\frac{i}{2}}(E_r y E_r)\right]\zeta_1 \mid \zeta_2\right)\end{aligned}$$

Hence

$$E_r x E_r = (\sigma_{-\frac{i}{2}} + e^s\sigma_{\frac{i}{2}})(E_r y E_r) = (e^{-\frac{s}{2}}\sigma_{-\frac{i}{2}} + e^{\frac{s}{2}}\sigma_{\frac{i}{2}})(e^{\frac{s}{2}}E_r y E_r).$$

By Lemma 1.21 we can invert the operator on $e^{s/2}E_r y E_r$ to obtain

$$e^{\frac{s}{2}}E_r y E_r = \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}}\Delta^{it}E_r x E_r\Delta^{-it}\ dt.$$

So letting $r \to \infty$ yields the lemma. □

To simplify notation we define $\rho_s \colon \mathcal{B}(\mathfrak{H}) \to \mathcal{B}(\mathfrak{H})$ for $s \in \mathbb{R}$ by:

$$\rho_s(x) := \int_{-\infty}^{\infty} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}}\Delta^{it}x\Delta^{-it}\ dt.$$

We finally are able to reap the benefit our labors.

Theorem 1.24. *Let $\mathfrak{A}$ be a left Hilbert algebra with modular operator Δ and modular conjugation J.*

(i)

$$\begin{aligned}J\mathcal{R}_l(\mathfrak{A})J &= \mathcal{R}_l(\mathfrak{A})';\\ J\mathcal{R}_l(\mathfrak{A})'J &= \mathcal{R}_l(\mathfrak{A});\\ \Delta^{it}\mathcal{R}_l(\mathfrak{A})\Delta^{-it} &= \mathcal{R}_l(\mathfrak{A});\\ \Delta^{it}\mathcal{R}_l(\mathfrak{A})'\Delta^{-it} &= \mathcal{R}_l(\mathfrak{A}), \qquad t \in \mathbb{R}.\end{aligned}$$

(ii) *The one parameter unitary group $\{\Delta^{it} \colon t \in \mathbb{R}\}$ acts on $\mathfrak{A}''$ and $\mathfrak{A}'$ as automorphisms and the modular conjugation J maps $\mathfrak{A}''$ (resp. $\mathfrak{A}'$) onto $\mathfrak{A}'$ (resp. $\mathfrak{A}''$) anti-isomorphically in the sense that*

$$J(\xi\eta) = (J\eta)(J\xi), \qquad \xi, \eta \in \mathfrak{A}''.$$

Proof. Lemma 1.20 implies that the hypothesis of Lemma 1.23 are satisfied for $x = \pi_r(\eta)$ and $y = J\pi_l((\Delta + e^s)^{-1})\eta)^*J$, hence

$$e^{\frac{s}{2}}J\pi_l((\Delta + e^s)^{-1}\eta)^*J = \rho_s(\pi_r(\eta)), \qquad \eta \in \mathfrak{A}',\ s \in \mathbb{R}.$$

Thus for $\zeta \in \mathfrak{A}'$ we have

$$\begin{aligned} J\rho_s(\pi_r(\eta))J\zeta &= e^{\frac{s}{2}}\pi_l((\Delta+e^s)^{-1}\eta)^*\zeta = e^{\frac{s}{2}}(S(\Delta+e^s)^{-1}\eta)\zeta \\ &= e^{\frac{s}{2}}\pi_r(\zeta)J\Delta^{\frac{1}{2}}(\Delta+e^s)^{-1}\eta = \pi_r(\zeta)J\int_{-\infty}^{\infty}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\Delta^{it}\eta\ dt, \end{aligned}$$

where we have applied Lemma 1.22 to obtain the last equality. Recalling the definition of ρ_s we see that this implies

$$\int_{-\infty}^{\infty}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\left(J\Delta^{it}\pi_r(\eta)\Delta^{-it}J\zeta - \pi_r(\zeta)J\Delta^{it}\eta\right)\ dt = 0$$

By the uniqueness of the Fourier transform we conclude for all $t \in \mathbb{R}$ that

$$\frac{1}{e^{\pi t}+e^{-\pi t}}\left(J\Delta^{it}\pi_r(\eta)\Delta^{-it}J\zeta - \pi_r(\zeta)J\Delta^{it}\eta\right) = 0,$$

or

$$J\Delta^{it}\pi_r(\eta)\Delta^{-it}J\zeta = \pi_r(\zeta)J\Delta^{it}\eta.$$

Hence $\|\pi_r(\zeta)J\Delta^{it}\eta\| = \|J\Delta^{it}\pi_r(\eta)\Delta^{-it}J\zeta\| \leq \|J\Delta^{it}\pi_r(\eta)\Delta^{-it}J\|\|\zeta\|$, so that $J\Delta^{it}\eta$ is left bounded and

$$\pi_l(J\Delta^{it}\eta) = J\Delta^{it}\pi_r(\eta)\Delta^{-it}J. \tag{2}$$

Now, since $F = \Delta^{1/2}J$, $J\mathfrak{D}^\flat = \mathfrak{D}(\Delta^{1/2}) = \mathfrak{D}^\sharp$, $J\Delta^{it}\eta \in \mathfrak{A}''$. Setting $t = 0$ we get $J\mathfrak{A}' \subset \mathfrak{A}''$ and

$$\pi_l(J\eta) = J\pi_r(\eta)J, \qquad \eta \in \mathfrak{A}'.$$

This implies that J is an anti-homomorphism on $\mathfrak{A}'$:

$$(J\xi)(J\eta) = \pi_l(J\xi)J\eta = J\pi_r(\xi)JJ\eta = J\pi_r(\xi)\eta = J(\eta\xi),$$

for $\xi, \eta \in \mathfrak{A}'$. By symmetry we get $J\Delta^{it}\xi \in \mathfrak{A}'$, $J\mathfrak{A}'' \subset \mathfrak{A}'$, and $\pi_r(J\xi) = J\pi_l(\xi)J$ for $\xi \in \mathfrak{A}''$. Thus we have

$$J\mathfrak{A}'' = \mathfrak{A}', \qquad J\mathfrak{A}' = \mathfrak{A}''.$$

Consequently $J\Delta^{it}\eta \in \mathfrak{A}''$ implies $\Delta^{it}\eta \in \mathfrak{A}'$ or $\Delta^{it}\mathfrak{A}' \subset \mathfrak{A}'$. We get equality by considering $-t$. Symmetry yeilds $\Delta^{it}\mathfrak{A}'' = \mathfrak{A}''$. Thus (ii) holds.

Part (i) then follows easily from formula (2), part (ii), and symmetry. □

We conclude with a few results to help us with Tomita algebras in the following section.

Proposition 1.25. *If $\mathfrak{A}$ is a left Hilbert algebra, then every central element $a \in \mathcal{R}_l(\mathfrak{A})$ leaves $\mathfrak{D}^\sharp$ and $\mathfrak{D}^\flat$ invariant and*

$$\begin{aligned} (a\xi)^\sharp &= a^*\xi^\sharp, \qquad \xi \in \mathfrak{D}^\sharp; \\ (a\eta)^\flat &= a^*\eta^\flat, \qquad \eta \in \mathfrak{D}^\flat. \end{aligned}$$

Furthermore,

$$JaJ = a^*, \qquad \Delta^{it}a\Delta^{-it} = a, \qquad t \in \mathbb{R}.$$

Proof. Without loss of generality $\mathfrak{A}$ is full. Since $a \in \mathcal{R}_l(\mathfrak{A}) \cap \mathcal{R}_l(\mathfrak{A})'$, Lemmas 1.11 and 1.11′ imply $\mathfrak{n}_r, \mathfrak{n}_r^*, \mathfrak{n}_l$, and $\mathfrak{n}_l^*$ are invariant under a. Thus part (ii) of Lemmas 1.17 and 1.17′ imply $a\pi_r(\mathfrak{A}') \subset \pi_r(\mathfrak{A}')$ and $a\pi_l(\mathfrak{A}) \subset \pi_l(\mathfrak{A})$. Appealing to Lemmas 1.11 and 1.11′ again tells us that this means $a\mathfrak{A}' \subset \mathfrak{A}'$ and $a\mathfrak{A} \subset \mathfrak{A}$. Furthermore, if $\eta \in \mathfrak{A}'$ then

$$\pi_r((a\eta)^\flat) = \pi_r(a\eta)^* = (a\pi_r(\eta))^* = \pi_r(\eta)^*a^* = a^*\pi_r(\eta^\flat) = \pi_r(a^*\eta^\flat),$$

so that $(a\eta)^\flat = a^*\eta^\flat$. The density of $\mathfrak{A}'$ in $\mathfrak{D}^\flat$ implies that $\mathfrak{D}^\flat$ is invariant under a and that the desired formula holds. A similar arguments shows this is true of $\mathfrak{D}^\sharp$ as well and $(a\xi)^\sharp = a^*\xi^\sharp$.

Suppose $a = u$ is a unitary, then we obtain $u\mathfrak{A} = \mathfrak{A}$ and $u\mathfrak{A}' = \mathfrak{A}'$ from the above work. Also, we have $Su\xi = u^*S\xi$ for $\xi \in \mathfrak{D}^\sharp$. Hence $uSu = S$ and thus

$$J\Delta^{\frac{1}{2}} = uJ\Delta^{\frac{1}{2}}u = uJuu^*\Delta^{\frac{1}{2}}u.$$

The uniqueness of the polar decomposition implies $J = uJu$ and $\Delta^{1/2} = u^*\Delta^{1/2}u$. That is, $JuJ = u^*$ and $\Delta^{it}u\Delta^{-it} = u$. Since $\mathcal{R}_l(\mathfrak{A}) \cap \mathcal{R}_l(\mathfrak{A})'$ is spanned linearly by unitaries, this holds for general a. □

Lastly, we present some topological results regarding π_l.

Proposition 1.26. *Let $\mathfrak{A}$ be a full left Hilbert algebra with completion $\mathfrak{H}$ and the associated algebra $\mathfrak{B}$ of all left bounded operators.*

(i) *The map $\pi_l\colon \mathfrak{A} \to \mathcal{R}_l(\mathfrak{A})$ is closed with respect to the $\|\cdot\|_\sharp$-norm topology in $\mathfrak{D}^\sharp$ and the σ-strong* topology in $\mathcal{R}_l(\mathfrak{A})$.*

(ii) *The map $\pi_l\colon \mathfrak{B} \to \mathcal{R}_l(\mathfrak{A})$ is closed with respect to the $\|\cdot\|$-norm topology in $\mathfrak{H}$ and the σ-strong topology in $\mathcal{R}_l(\mathfrak{A})$.*

Proof.

(i): Let $\{\xi_\alpha\} \subset \mathfrak{A}$ be a net converging to ξ with respect to $\|\cdot\|_\sharp$ such that $\{\pi_l(\xi_\alpha)\}$ converges to $x \in \mathcal{R}_l(\mathfrak{A})$ σ-strongly*. Then for any $\eta \in \mathfrak{A}$ we have

$$\pi_r(\eta)\xi = \lim_\alpha \pi_r(\eta)\xi_\alpha = \lim_\alpha \pi_l(\xi_\alpha)\eta = x\eta;$$
$$\pi_r(\eta)\xi^\sharp = \lim_\alpha \pi_r(\eta)\xi_\alpha^\sharp = \lim_\alpha \pi_l(\xi_\alpha)^*\eta = x^*\eta.$$

Hence ξ is left bounded and $x = \pi_l(\xi)$.

(ii): The prove is the same as part (i), ignoring any statements about $\sharp$ and $*$. □

Lemma 1.27. *Let $\{x_\alpha\}$ be a net in $\mathcal{B}(\mathfrak{H})$ and $x \in \mathcal{B}(\mathfrak{H})$. If there exist dense subsets $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathfrak{H}$ such that $\lim_\alpha x_\alpha\xi = x\xi$, $\xi \in \mathfrak{M}$, and $\lim_\alpha x_i^*\eta = x\eta$, $\eta \in \mathfrak{N}$, in norm, then for any continuous bounded function f on $[0,\infty)$ we have the convergence:*

$$\lim_\alpha f(x_\alpha^* x_\alpha) = f(x^*x), \qquad \lim_\alpha f(x_\alpha x_\alpha^*) = f(xx^*)$$

in the strong operator topology.

Proof. Let $g(t) = f(t^2)$, $t \in \mathbb{R}$, and h_α and h the self-adjoint operators on $\mathfrak{H} \oplus \mathfrak{H}$ give by the matrices:

$$h_\alpha := \begin{pmatrix} 0 & x_\alpha^* \\ x_\alpha & 0 \end{pmatrix} \qquad \text{and} \qquad h := \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}.$$

Then $\lim_\alpha h_\alpha\xi = h\xi$ for any $\xi \in \mathfrak{M} \oplus \mathfrak{N}$. Then by Lemma II.4.6 in Takesaki [2] we know $\{g(h_\alpha)\}$ converges to $g(h)$ in the strong operator topology. Since

$$g\left[\begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix}\right] = \begin{pmatrix} f(y^*y) & 0 \\ 0 & f(yy^*) \end{pmatrix}, \qquad y \in \mathcal{B}(\mathfrak{H}),$$

this gives precisely the desired result. □

Theorem 1.28. *Let $\mathfrak{A}$ be a left Hilbert algebra with completion $\mathfrak{H}$.*

(i) *If $\xi \in \mathfrak{A}''$, then there exists a sequence $\{\xi_n\} \subset \mathfrak{A}$ such that*

$$\lim_{n\to\infty} \|\xi - \xi_n\|_\sharp = 0 \qquad \textit{and} \qquad \|\pi_l(\xi_n)\| \le \|\pi_l(\xi)\|.$$

Hence $\{\pi_l(\xi_n)\}$ converges to $\pi_l(\xi)$ in the strong* operator topology.

(ii) *If $\xi \in \mathfrak{H}$ is left bounded, then there exists a sequence $\{\xi_n\} \subset \mathfrak{A}$ such that*

$$\lim_{n\to\infty} \|\xi - \xi_n\| = 0 \qquad \textit{and} \qquad \|\pi_l(\xi_n)\| \le \|\pi_l(\xi)\|.$$

Hence $\{\pi_l(\xi_n)\}$ converges to $\pi_l(\xi)$ in the strong operator topology.

Proof.

(i): We can assume $\|\pi_l(\xi)\| = 1$. Since $\xi \in \mathfrak{D}^\sharp$ we can find a sequence $\{\zeta_n\} \subset \mathfrak{A}$ such that $\|\zeta_n - \xi\|_\sharp \to 0$ as $n \to \infty$. Set $x = \pi_l(\xi)$ and $x_n = \pi_l(\zeta_n)$. For every $\eta \in \mathfrak{A}'$ we have the convergence:

$$\|x\eta - x_n\eta\| \to 0 \qquad \text{and} \qquad \|x^*\eta - x_n^*\eta\| \to 0$$

since η is right bounded. Hence the preceding lemma applies. Consider the function f defined on $[0,\infty)$ by

$$f(t) := \begin{cases} 1 & 0 \le t \le 1 \\ t^{-\frac{1}{2}} & t > 1. \end{cases}$$

The lemma gives us that $f(x_n^*x_n) \to f(x^*x) = 1$ and $f(x_nx_n^*) \to f(xx^*) = 1$ strongly. Set $\xi_n'' = f(x_nx_n^*)\zeta_n$. Then the dualized version of formula (1) shows that $\xi_n'' \in \mathfrak{A}''$ and $(\xi_n'')^\sharp = f(x_n^*x_n)\zeta_n^\sharp$. Furthermore, we have

$$\pi_l(\xi_n'') = f(x_nx_n^*)x_n, \qquad \pi_l(\xi_n'')^* = f(x_n^*x_n)x_n^*.$$

Now, we have

$$\begin{aligned}\|\xi_n'' - \xi\| &\leq \|f(x_nx_n^*)(\zeta_n - \xi)\| + \|f(x_nx_n^*)\xi - \xi\| \\ &\leq \|\zeta_n - \xi\| + \|(f(x_nx_n^*) - 1)\xi\| \to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Similarly, we can show that $\|(\xi_n'')^\sharp - \xi\| \to 0$ and thus $\|\xi_n'' - \xi\|_\sharp \to 0$. Moreover, we have

$$\|\pi_l(\xi_n'')\|^2 = \|\pi_l(\xi_n'')\pi_l(\xi_n'')^*\| = \|f(x_nx_n^*)x_nx_n^*f(x_nx_n^*)\| \leq \sup_{t\geq 1}|tf(t)^2| = 1.$$

Now, approximate f by polynomials p_n on $[0,\infty)$ so that

$$|p_n(t) - f(t)| \leq \frac{1}{n}\min\left\{\frac{1}{\|x_n\|}, \frac{1}{\|\zeta_n\|}, \frac{1}{\|\zeta_n^\sharp\|}\right\}, \qquad 0 \leq t \leq \|x_n\|^2.$$

Then set $\xi_n' = p_n(\zeta_n\zeta_n^\sharp)\zeta_n$. Then $\xi_n' \in \mathfrak{A}$ and $(\xi_n')^\sharp = p_n(\zeta_n^\sharp\zeta_n)\zeta_n^\sharp$. We get that

$$\begin{aligned}\|\xi_n' - \xi_n''\| &\leq \|p_n(x_nx_n^*) - f(x_nx_n^*)\|\|\zeta_n\| \leq \frac{1}{n}; \\ \|(\xi_n')^\sharp - (\xi_n'')^\sharp\| &\leq \|p_n(x_n^*x_n) - f(x_n^*x_n)\|\|\zeta_n^\sharp\| \leq \frac{1}{n}.\end{aligned}$$

Thus $\|\xi_n' - \xi_n''\|_\sharp \to 0$ as $n \to \infty$. Furthermore, we compute

$$\|\pi_l(\xi_n')\| = \|p_n(x_nx_n^*)x_n\| \leq \|p_n(x_n^*x_n) - f(x_nx_n^*)\|\|x_n\| + \|f(x_nx_n^*)x_n\| \leq \frac{1}{n} + 1.$$

Setting $\xi_n := \left(1 + \frac{1}{n}\right)^{-1}\xi_n''$, we finally obtain a sequence in $\mathfrak{A}$ which converges to ξ in $\mathfrak{D}^\sharp$ and such that $\|\pi_l(\xi_n)\| \leq 1$ for all n. To see that $\{\pi_l(\xi_n)\}$ converges to $\pi_l(\xi)$ in the strong* topology we note that for each $\eta \in \mathfrak{A}'$ we have

$$\begin{aligned}\pi_l(\xi_n)\eta = \pi_r(\eta)\xi_n &\overset{n\to\infty}{\longrightarrow} \pi_r(\eta)\xi = \pi_l(\xi)\eta; \\ \pi_l(\xi_n)^*\eta = \pi_r(\eta)\xi_n^\sharp &\overset{n\to\infty}{\longrightarrow} \pi_r(\eta)\xi^\sharp = \pi_l(\xi)^*\eta.\end{aligned}$$

The uniform bound on the norms of the $\pi_l(\xi_n)$ and the density of $\mathfrak{A}'$ then imply convergence in the strong* topology.

(ii): Suppose $\xi \in \mathfrak{B}$. Without loss of generality we have $\|\pi_l(\xi)\| = 1$. Let $\pi_l(\xi) = uh$ be the polar decomposition and set $\zeta = u^*\xi$. Then $\zeta \in \mathfrak{B}$ and $\pi_l(\zeta) = h$ is self-adjoint, so that $\zeta \in \mathfrak{A}''$ and $\zeta = \zeta^\sharp$. Using (i) we choose a sequence $\{\zeta_n\} \subset \mathfrak{A}$ such that $\|\zeta - \zeta_n\| < \frac{1}{2n}$ and $\|\pi_l(\zeta_n)\| \leq 1$. Since $u \in \mathcal{R}_l(\mathfrak{A})$, there exists, by Kaplanksy's density theorem, a sequence $\{\eta_n\}$ in $\mathfrak{A}$ such that

$$\|\pi_l(\eta_n)\| \leq 1 \qquad \|\eta_n\zeta_n - u\zeta_n\| < \frac{1}{2n}.$$

Set $\xi_n = \eta_n\zeta_n \in \mathfrak{A}$ to get

$$\begin{aligned}\|\xi_n - \xi\| \leq \|\eta_n\zeta_n - u\zeta_n\| + \|u(\zeta_n - \zeta)\| &< \frac{1}{n}; \\ \|\pi_l(\xi_n)\| = \|\pi_l(\eta_n)\pi_l(\zeta_n)\| &\leq 1.\end{aligned}$$

Using the argument at the end of part (i), we see that $\{\pi_l(\xi_n)\}$ converges strongly to $\pi_l(\xi)$. □

2. Tomita Algebras

The idea in this section is to produce a "self-adjoint" subalgebra $\mathfrak{A}_0$ of both $\mathfrak{A}$ and $\mathfrak{A}'$ such that $\mathfrak{A}'' = \mathfrak{A}_0''$ and $\mathfrak{A}' = \mathfrak{A}_0'$ on which $\Delta^{i\alpha}$ is an entire function.

Definition 2.1. A left Hilbet algebra $\mathfrak{A}$ is called a **Tomita algebra** if $\mathfrak{A}$ admits a complex one parameter group $\{U(\alpha)\colon \alpha \in C\}$ of automorphisms, not necessarily $*$-preserving, with the following properties:

a. The function $\mathbb{C} \ni \alpha \mapsto (U(\alpha)\xi \mid \eta)$ is entire;
b. $(U(\alpha)\xi)^\sharp = U(\overline{\alpha})\xi^\sharp$;

c. $(U(\alpha)\xi \mid \eta) = (\xi \mid U(-\overline{\alpha})\eta)$, $\alpha \in \mathbb{C}$, $\xi, \eta \in \mathfrak{A}$;
d. $(\xi^\sharp \mid \eta^\sharp) = (U(-i)\eta \mid \xi)$.

The group $\{U(\alpha) \colon \alpha \in \mathbb{C}\}$ is called the **modular automorphism group** of $\mathfrak{A}$.

Theorem 2.2.

(i) *Given a full left Hilbert algebra $\mathfrak{A}$ with modular operator Δ, if we set*

$$\mathfrak{A}_0 := \left\{ \xi \in \bigcap_{n\in\mathbb{Z}} \mathfrak{D}(\Delta^n) \colon \Delta^n \xi \in \mathfrak{A},\ n \in \mathbb{Z} \right\}, \tag{3}$$

then $\mathfrak{A}_0$ is a Tomita algebra with respect to $\{\Delta^{i\alpha} \colon \alpha \in \mathbb{C}\}$ such that

$$\mathfrak{A}_0'' = \mathfrak{A}, \qquad \mathfrak{A}_0' = \mathfrak{A}, \qquad \text{and} \qquad J\mathfrak{A}_0 = \mathfrak{A}_0.$$

Hence, in particular we have

$$\mathcal{R}_l(\mathfrak{A}_0) = \mathcal{R}_l(\mathfrak{A}) \qquad \text{and} \qquad \mathcal{R}_r(\mathfrak{A}_0) = \mathcal{R}_l(\mathfrak{A})'$$

(ii) *If $\mathfrak{A}$ is a Tomita algebra, then with the new involution:*

$$\xi^\flat := U(-i)\xi^\sharp,\ \xi \in \mathfrak{A},$$

$\mathfrak{A}$ is a right Hilbert algebra and

$$\mathcal{R}_l(\mathfrak{A})' = \mathcal{R}_r(\mathfrak{A}).$$

Furthermore, the modular operator Δ is the closure of $U(-i)$.

We shall require some lemmas, the first of which characterizes when a vector belongs to the domain of a power of a self-adjoint operator.

Lemma 2.3. *Let H be a non-singular self-adjoint positive operator on a Hilbert space $\mathfrak{H}$. For fixed $\alpha \in \mathbb{R}$ and $\xi \in \mathfrak{A}$, the following two conditions are equivalent:*

(i) *ξ belongs to the domain $\mathfrak{D}(H^\alpha)$ of H^α;*
(ii) *The $\mathfrak{H}$-valued function: $\mathbb{R} \ni t \mapsto H^{it}\xi \in \mathfrak{H}$ can be extended to an $\mathfrak{H}$-valued function: $\mathbb{D}_\alpha \ni \omega \mapsto \xi(\omega) \in \mathfrak{H}$ such that ξ is continuous and bounded on the closure $\overline{\mathbb{D}_\alpha}$ and holomorphic in $\mathbb{D}_\alpha$, where $\mathbb{D}_\alpha$ is the horizontal strip bounded by $\mathbb{R}$ and $\mathbb{R} - i\alpha$.*

Proof. By considering H^{-1} if necessary, we may assume $\alpha > 0$.

$(i) \Rightarrow (ii)$: If $\omega = t - is \in \mathbb{D}_\alpha$, then $\mathfrak{D}(H^{i\omega}) = \mathfrak{D}(H^s)$. The inequality

$$\|H^{i\omega}\xi\| = \|H^s\xi\| \leq \|(1+H)^s\xi\| \leq \|(1+H)^\alpha\xi\|,$$

shows that $\xi(\omega) := H^{i\omega}\xi$ is bounded and continuous on $\mathbb{D}_\alpha$. Let

$$H = \int_0^\infty \lambda dE(\lambda)$$

be the spectral decomposition and set

$$\mathfrak{M} := \bigcup_{n=1}^\infty [E([0,n]) - E([0,1/n])]\mathfrak{H}.$$

For each $\eta \in \mathfrak{M}$, setting $\eta(\omega) := H^{i\omega}\eta$, $\omega \in \mathbb{C}$, we obtain an $\mathfrak{H}$-valued function $\eta(\cdot)$. From the integral representation,

$$(H^{i\omega}\eta \mid \zeta) = \int_{1/n}^n \lambda^{i\omega} d(E(\lambda)\eta \mid \zeta), \qquad \zeta \in \mathfrak{H},$$

with a sufficiently large n, $\eta(\cdot)$ is entire. Now, for every $\eta \in \mathfrak{M}$,

$$(\xi(\omega) \mid \eta) = (H^{i\omega}\xi \mid \eta) = (\xi \mid H^{-i\bar{\omega}}\eta) = \overline{(\eta(-i\bar{\omega}) \mid \xi)}.$$

Hence the function $\mathbb{D}_\alpha \ni \omega \mapsto (\xi(\omega) \mid \eta)$ is holomorphic (since $\eta(\cdot)$ is entire) for every $\eta \in \mathfrak{M}$. Noting that $\mathfrak{M}$ is dense in $\mathfrak{H}$ we see that $\xi(\cdot)$ is holomorphic on $\mathfrak{D}_\alpha$.

$(ii) \Rightarrow (i)$: Suppose $\xi(t) := H^{it}\xi$ can be extended to $\mathbb{D}_\alpha$. From the previous argument, we know each $\eta \in\in \mathfrak{D}(H^\alpha)$ gives rise to a bounded continuous function $\eta(\omega) = H^{i\omega}\eta$ on $\overline{\mathbb{D}_\alpha}$ that is holomorphic in $\mathbb{D}_\alpha$. Consider the two functions on $\overline{\mathbb{D}_\alpha}$:

$$\omega \mapsto (\xi(\omega) \mid \eta) \qquad \text{and} \qquad \omega \mapsto (\xi \mid H^{i\bar{\omega}}\eta).$$

Then agree when $\omega = t \in \mathbb{R}$ since there $\xi(t) = H^{it}\xi$. Thus the analyticity implies the agree on the entier strip $\overline{\mathbb{D}_\alpha}$; hence

$$(\xi(\omega) \mid \eta) = (\xi \mid H^{-i\bar{\omega}}\eta), \qquad \eta \in \mathfrak{D}(H^\alpha).$$

Setting $\omega = -i\alpha$ we get that

$$(\xi \mid H^\alpha \eta) = (\xi(-i\alpha) \mid \eta), \qquad \eta \in \mathfrak{D}(H^\alpha),$$

which shows that $\xi \in \mathfrak{D}((H^\alpha)^*) = \mathfrak{D}(H^\alpha)$ and that $H^\alpha\xi = \xi(-i\alpha)$. □

Lemma 2.4. *Let K be a compact convex subset of a locally convex vector space E. If a function $x\colon \mathbb{R} \ni t \mapsto x(t) \in E$ is continuous and takes values in K, then the Bochner integral for each $r > 0$*

$$x_r := \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-rt^2} x(t)\ dr$$

belongs to K and $\lim_{r\to\infty} x_r = x(0)$.

Proof. Since $e^{-rt^2} > 0$ and $(r/\pi)^{1/2} \int_{\mathbb{R}} e^{-rt^2}\ dt = 1$, $x_r \in K$ by the compactness and convexity of K. If p is a continuous semi-norm on E, then

$$p\left(\sqrt{\frac{r}{\pi}}\int_R e^{-rt^2} x(t)\ dt - x(0)\right) = p\left(\sqrt{\frac{r}{\pi}}\int_R e^{-rt^2}[x(t) - x(0)]\ dt\right) \leq \sqrt{\frac{r}{\pi}}\int_R e^{-rt^2} p[x(t) - x(0)]\ dt.$$

On the other hand, $\{(r/\pi)^{1/2} e^{-rt^2}\}_{r>0}$ is an approximate identity; that is, integrating against it with a continuous bounded function on $\mathbb{R}$ and letting $r \to \infty$ is equivalent to evaluating at zero. Hence the last expression in the above inequality converges to zero as $r \to \infty$. As p was an arbitrary continuous seminorm, we conclude that x_r converges to $x(0)$ in the locally convex topology. □

Proof of Theorem 2.2.

(i): Suppose that $\mathfrak{A}$ is a left Hilbert algebra nd $\mathfrak{A}_0$ is defined in by (3). Hence $\xi = \Delta^0\xi \in \mathfrak{A}$ for all ξ, so that $\mathfrak{A}_0 \subset \mathfrak{A}$. If $\xi \in \mathfrak{A}_0$, then $\xi = (1 + \Delta^{-1})^{-1}(1 + \Delta^{-1})\xi \in (1 + \Delta^{-1})^{-1}\mathfrak{A}$, so that ξ belongs to $\mathfrak{A}'$ by Lemma **??**.(ii). Hence $\mathfrak{A}_0 \subset \mathfrak{A} \cap \mathfrak{A}'$. It also follows that $J\mathfrak{A}_0 = \mathfrak{A}_0$. Indeed, if $\xi \in \mathfrak{A}_0$ then $\Delta^{-n}\xi \in \mathfrak{A}_0$ for all $n \in \mathbb{Z}$ (by definition of $\mathfrak{A}_0$. But then $\Delta^{-n}\xi \in \mathfrak{A}'$ and so by Theorem 1.24.(ii), $J(\Delta^{-n}\xi) \in \mathfrak{A}$. Thus $\Delta^n(J\xi) = J(\Delta^{-n}\xi) \in \mathfrak{A}$ for all $n \in \mathbb{Z}$ so that $J\xi \in \mathfrak{A}_0$.

Suppose $\xi \in \mathfrak{A}_0$. Then by Lemma 2.3, the function $\mathbb{C} \ni \alpha \mapsto \Delta^{i\alpha}\xi \in \mathfrak{H}$ is entire. We want to show that $\Delta^{i\alpha}\xi \in \mathfrak{A}_0$ for $\alpha \in \mathbb{C}$ (so that $U(\alpha) = \Delta^{i\alpha}$ would be a candidate for the modular automorphism group). Let $\alpha = r + is$ and $n = \lfloor s \rfloor$. By Theorem 1.24, Δ^{ir} acts on $\mathfrak{A}'' = \mathfrak{A}$ as an automorphism and hence $\Delta^m(\Delta^{ir}\xi) = \Delta^{ir}(\Delta^m\xi) \in \mathfrak{A}$ for all $m \in \mathbb{Z}$ since $\xi \in \mathfrak{A}_0$. Hence $\Delta^{ir}\xi \in \mathfrak{A}_0$. Similarly, $\Delta^m\xi \in \mathfrak{A}_0$ for all $m \in \mathbb{Z}$. Furthermore, for $m = -n, -n-1$ we get that $\Delta^m\xi \in \mathfrak{A}$ so that $\pi_l(\Delta^m\xi)$ is bounded. Now, for any $\eta \in \mathfrak{A}'$, we have for each $m \in \mathbb{Z}$

$$\sup_{t\in\mathbb{R}} \|\pi_r(\eta)\Delta^{it+m}\xi\| \leq \|\pi_l(\Delta^m\xi)\|\|\eta\|.$$

So by the Phragmém-Lindeolöf theorem, we have

$$\|\pi_r(\eta)\Delta^{i\alpha}\xi\| \leq \max\left\{\|\pi_l(\Delta^{-n}\xi)\|, \|\pi_l(\Delta^{-n-1}\xi)\|\right\}\|\eta\|.$$

Hence $\Delta^{i\alpha}\xi$ is left bounded. But also $\Delta^{i\alpha}\xi \in \mathfrak{D}(\Delta^m)$ for all $m \in \mathbb{Z}$. In particular this is true for $m = 1$. Thus $\Delta^{i\alpha}\xi$ is both left bounded and contained in $\mathfrak{D}(\Delta) = \mathfrak{D}^\sharp$: the intersection of these two sets is precisely $\mathfrak{A}'' = \mathfrak{A}$. Hence $\Delta^{i\alpha}\xi \in \mathfrak{A}$, but since α was arbitrary, acting on this by Δ^m for any $m \in \mathbb{Z}$ leaves it in $\mathfrak{A}$, ergo $\Delta^{i\alpha}\xi \in \mathfrak{A}_0$. So the set $\{\Delta^{i\alpha}\colon \alpha \in \mathbb{C}\}$ leaves $\mathfrak{A}_0$ globally invariant.

We next show the $\Delta^{i\alpha}$ are homomorphisms. Theorem 1.24 tells us they are when $\alpha \in \mathbb{R}$, so given $\xi, \eta \in \mathfrak{A}_0$ we know $\Delta^{it}(\xi\eta) = (\Delta^{it}\xi)(\Delta^{it}\eta)$. Hence the functions $\alpha \mapsto \Delta^{i\alpha}(\xi\eta)$ and $\alpha \mapsto (\Delta^{i\alpha}\xi)(\Delta^{i\alpha}\eta)$ agree on $\mathbb{R}$ and so by the uniqueness of the holomorphic extension are equal everywhere and hence $\Delta^{i\alpha}$ is multiplicative. Hence $\{\Delta^{i\alpha}\colon \alpha \in \mathbb{C}\}$ is a one parameter group of automorphisms of $\mathfrak{A}_0$, and given what we already know of Δ it is easy to see that $\{\mathfrak{A}_0, \Delta^{i\alpha} : \alpha \in \mathbb{C}\}$ satisfies the conditions in the definition of a Tomita algebra.

Next we verify $\mathfrak{A}_0'\mathfrak{A}$. Fix $\xi \in \mathfrak{A}$ and $r > 0$ and set

$$\xi_r := \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-rt^2} \Delta^{it} \xi \; dt.$$

It follows that $\xi_r \in \mathfrak{D}(\Delta^{i\alpha})$, $\alpha \in \mathbb{C}$, and that after a change of variables we have

$$\Delta^{i\alpha} \xi_r = \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-r(t-\alpha)^2} \Delta^{it} \xi \; dt.$$

Then for each $\eta \in \mathfrak{A}'$ we have

$$\begin{aligned} \pi_r(\eta)\Delta^{i\alpha}\xi_r &= \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-r(t-\alpha)^2} \pi_r(\eta) \Delta^{it} \xi \; dt \\ &= \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-r(t-\alpha)^2} \pi_l(\Delta^{it}\xi)\eta \; dt = \sqrt{\frac{r}{\pi}} \left(\int_{\mathbb{R}} e^{-r(t-\alpha)^2} \Delta^{it} \pi_l(\xi) \Delta^{-it} \; dt \right) \eta, \end{aligned}$$

so that $\Delta^{i\alpha}\xi_r$ is left bounded by Lemma 2.4 with $K =$??? and $E =$???. As before, this implies $\Delta^{i\alpha}\xi_r \in \mathfrak{A}$ for every $\alpha \in \mathbb{C}$. Consequently $\xi_r \in \mathfrak{A}_0$. Lemma 2.4 also gives us that

$$\lim_{r\to\infty} \xi_r = \xi, \qquad \lim_{r\to\infty} \Delta^{\frac{1}{2}}\xi_r = \lim_{r\to\infty} (\Delta^{\frac{1}{2}}\xi)_r = \Delta^{\frac{1}{2}}\xi,$$

so that ξ is approximated by ξ_r in the $\sharp$-norm. Since $\xi \in \mathfrak{A}$ was arbitrary, this implies that $\mathfrak{A}_0$ is a core of $\Delta^{1/2}$. Furthermore, $\mathfrak{A}_0 = J\mathfrak{A}_0$ is dense in $J\mathfrak{D}^\sharp = \mathfrak{D}^\flat$, so that $\mathfrak{A}_0$ is also a core of $\mathfrak{D}^{-1/2}$. Therefore, the closure of the $\sharp$-operation in $\mathfrak{A}_0$ agrees with S and that of the $\flat$-operation in $\mathfrak{A}_0$ agrees with F.

So if $\xi \in \mathfrak{A}$, then $\pi_l(\xi_r)$ converges to $\pi_l(\xi)$ σ-strongly as $r \to \infty$, so that if $\eta \in \mathfrak{A}$ is right bounded with respect to $\mathfrak{A}_0$ then

$$\|\pi_l(\xi)\eta\| = \lim_{r\to\infty} \|\pi_l(\xi_r)\eta\| \le \lim_{r\to\infty} c\|\xi_r\| \le c\|\xi\|,$$

which implies η is also right bounded with respect to $\mathfrak{A}$. Since being right bounded and being in $\mathfrak{D}^\flat$ means the same with respect to either $\mathfrak{A}$ or $\mathfrak{A}_0$ we have that $\mathfrak{A}_0' = \mathfrak{A}'$. Then being left bounded is the same with respect to either $\mathfrak{A}_0$ or $\mathfrak{A}$, and we already saw the closure of the involutions ahve the same domain so $\mathfrak{A}_0'' = \mathfrak{A}'' = \mathfrak{A}$.

(ii): Suppose $\{\mathcal{F}, U(\alpha) : \alpha \in \mathbb{C}\}$ is a Tomita algebra. Let Δ and J be the associated modular operator and the modular conjugation, and let $\mathfrak{H}$ be the completion of $\mathfrak{A}$. By the group property and Definition 2.1.(c) of $U(\alpha)$, if $t \in \mathbb{R}$ then for each $\xi \in \mathfrak{A}$ we have

$$\|U(t)\xi\|^2 = (U(t)\xi \mid U(t)\xi) = (\xi \mid U(-t)U(t)\xi) = (\xi \mid U(0)\xi) = \|\xi\|^2.$$

Hence $U(t)$ can be extended to a unitary on $\mathfrak{H}$, which is denoted by $U(t)$ again. From Definition 2.1.(a), we know $\alpha \mapsto (U(\alpha)\xi \mid \eta)$ is entire for any $\xi, \eta \in \mathfrak{A}$. It follows (non-trivially) that since $\mathfrak{A}$ is dense in $\mathfrak{H}$ the map: $\mathbb{C} \ni \alpha \mapsto U(\alpha)\xi$ is entire in norm for each $\xi \in \mathfrak{A}$. That is, there is a sequence of vectors $\{\xi_n\} \in \mathfrak{H}$ such that

$$U(\alpha)\xi = \sum_{n=0}^{\infty} \alpha^n \xi_n,$$

where the above sum converges in norm. Now, by Stone's theorem we can produce an infinitesimal generator H for $\{U(t)\}$. That is, H is a self-adjoint (possibly unbounded) operator such that if

$$H = \int_{\mathbb{R}} \lambda \; dE(\lambda)$$

is the spectral decomposition of H then

$$U(t) = \int_{\mathbb{R}} e^{i\lambda t} \; dE(\lambda),$$

for all $t \in \mathbb{R}$. We express this more concisely as $U(t) = \exp(itH)$. Moreover, $\mathfrak{D}(H)$ is precisely the set of vectors ξ for which the following limit exists:

$$\lim_{t\to 0} \frac{1}{t}(U(t) - 1)\xi = iH\xi.$$

Since $\alpha \mapsto U(\alpha)\xi$ is entire for $\xi \in \mathfrak{A}$ we know that $\mathfrak{A} \subset \mathfrak{D}(H)$.

Now,

$$\exp(i\alpha H) = \int_{\mathbb{R}} e^{i\lambda\alpha}\, dE(\lambda), \qquad \alpha \in \mathbb{C}$$

is closed and extends $U(\alpha)$ for each α. In particular, it extends $U(-in)$ so that $\mathfrak{A} \subset \mathfrak{D}(\exp(nH))$ for every $n \in \mathbb{Z}$. We want to show $\Delta = \exp H$, but it suffices so show they agree on $\mathfrak{A}$ and that $\mathfrak{A}$ is a common core. We already know $\mathfrak{A}$ is a core for Δ, so it remains to show they agree on $\mathfrak{A}$ and that $\mathfrak{A}$ is a core for $\exp H$. From Definition 2.1.(b), we know

$$SU(\alpha)\xi = U(\overline{\alpha})S\xi, \qquad \xi \in \mathfrak{A},\ \alpha \in \mathbb{C},$$

which means that

$$J\Delta\frac{1}{2}U(\alpha)\xi = U(\overline{\alpha})J\Delta^{\frac{1}{2}}\xi, \qquad \xi \in \mathfrak{H},\ \alpha \in \mathbb{C}.$$

In particular for $t \in \mathbb{R}$ we have $U(t)J\Delta^{\frac{1}{2}} = J\Delta^{\frac{1}{2}}U(t) = JU(t)U(-t)\Delta^{\frac{1}{2}}U(t)$; and hence the uniqueness of the polar decomposition implies

$$JU(t) = U(t)J \qquad \text{and} \qquad U(t)\Delta^{\frac{1}{2}} = \Delta^{\frac{1}{2}}U(t), \qquad t \in \mathbb{R}.$$

Thus implies the spectral projections of H and Δ commute, which is in turn equivalent to the commutativity of $\{U(t)\}$ and $\{\Delta^{it}\}$. Definition 2.1.(d) implies for $\eta \in \mathfrak{A}$ that $\xi \mapsto (\xi^\sharp \mid \eta) = (U(-i)\eta^\sharp \mid \xi)$ and thus is bounded. Consequently $\eta \in \mathfrak{D}^\flat$ and $\eta^\flat = U(-i)\eta^\sharp$ (recall $(F\eta \mid \xi) = (S\xi \mid \eta)$). Thus $\mathfrak{A} \subset \mathfrak{D}^\flat$ and

$$\Delta\xi = FS\xi = (\xi^\sharp)^\flat = U(-i)\xi = \exp(H)\xi, \qquad \xi \in \mathfrak{A}.$$

We next show that $\mathfrak{A}$ is a core for $\exp(sH)$, $s \in \mathbb{R}$. We first claim it suffices to show that $(1+K)\mathfrak{A}$ is dense in $\mathfrak{H}$. Indeed, let $\xi \in \mathfrak{D}(K)$ and take $(\xi_n) \subset (1+K)\mathfrak{A}$ converging to $(1+K)\xi$. Noting that K is positive (since H is self-adjoint) we have

$$\begin{aligned}\|\xi_n - \xi_m\|^2 &\leq +\|\xi_n - \xi_m\|^2 + \langle K(\xi_n - \xi_m), \xi_n - \xi_m\rangle \\ &= \langle (1+K)(\xi_n\xi_m), \xi_n - \xi_m\rangle \leq \|(1+K)(\xi_n - \xi_m)\|\|\xi_n - \xi_m\|,\end{aligned}$$

or $\|\xi_n - \xi_m\| \leq \|(1+K)(\xi_n - \xi_m)\|$. Hence (ξ_n) is a Cauchy sequence and we denote $\zeta = \lim_n \xi$. Consequently $\lim_n K\xi_n = \lim_n (1+K)\xi_n - \xi_n = (1+K)\xi - \zeta$. Since K is closed this implies $K\zeta = (1+K)\xi - \zeta$, or $(1+K)\zeta = (1+K)\xi$. But $1+K$ has dense range and consequently is injective. So it must be that $\xi = \zeta$ and hence (ξ_n) converges to ξ in the graph norm: $\|\cdot\|_K = \sqrt{\|\cdot\|^2 + \|K\cdot\|^2}$. Thus to show $\mathfrak{A}$ is a core for K it suffices to show $(1+K)\mathfrak{A}$ is dense $\mathfrak{H}$.

Applying Lemma 1.22 to $\{K^{it}\}$ and $s = 0$ we have

$$(K^{\frac{1}{2}} + K^{-\frac{1}{2}})^{-1} = \int_{\mathbb{R}} \frac{1}{e^{\pi t} + e^{-\pi t}} K^{it}\, dt = \int_{\mathbb{R}} \frac{1}{e^{\pi t} + e^{-\pi t}} U(st)\, dt.$$

Since $K = U(-is)$, we have for each $\xi \in \mathfrak{A}$:

$$\begin{aligned}\xi &= (1+K)^{-1}(1+K)\xi = (K^{-\frac{1}{2}} + K^{\frac{1}{2}})^{-1}K^{-\frac{1}{2}}(1+K)\xi \\ &= \int_{\mathbb{R}} \frac{1}{e^{\pi t} + e^{-\pi t}} U(st)K^{-\frac{1}{2}}(1+K)\xi\, dt \\ &= \int_{\mathbb{R}} (1+K)\frac{1}{e^{\pi t} + e^{-\pi t}} U\left(s\left(t - \frac{i}{2}\right)\right)\xi\, dt.\end{aligned}$$

Note that $\frac{1}{e^{\pi t}+e^{-\pi t}}U\left(s\left(t - \frac{i}{2}\right)\right)\xi \in \mathfrak{A}$. Approximating the above integral by Riemann sums shows that ξ is arbitrarily well approximated by $(1+K)\mathfrak{A}$.

Hence $\mathfrak{A}$ is a core for $K = \exp(sH)$ so that $\Delta = \exp(H)$. Consequently $\Delta^{it} = U(t)$ for $t \in \mathbb{R}$.

We can now show that $\mathfrak{A}$ is a right Hilbert algebra with involution $\flat$. Given $\xi, \eta, \zeta \in \mathfrak{A}$,

$$J(\xi\eta) = \Delta^{\frac{1}{2}} S(\xi\eta) = U\left(-\frac{i}{2}\right)(\eta^\sharp\xi^\sharp) = \left(U\left(-\frac{i}{2}\right)\eta^\sharp\right)\left(U\left(-\frac{i}{2}\right)\xi^\sharp\right) = (J\eta)(J\xi); \text{ and so}$$

$$\begin{aligned}(\xi\eta \mid \zeta) &= (J\zeta \mid J(\xi\eta)) = (J\zeta \mid (J\eta)(J\xi)) = ((J\eta)^\sharp J\zeta \mid J\xi) = \left(\xi \mid J[(\Delta^{-\frac{1}{2}}\eta)J\zeta]\right) \\ &= (\xi \mid \zeta(J\Delta^{-\frac{1}{2}}\eta)) = (\xi \mid \zeta(\Delta^{-1}\eta^\sharp)) = (\xi \mid \zeta(U(-i)\eta^\sharp)) = (\xi \mid \zeta\eta^\flat).\end{aligned}$$

Also, since $J = \Delta^{\frac{1}{2}} S = U\left(-\frac{i}{2}\right) S$ and since both S and $U(\alpha)$ map $\mathfrak{A}$ to $\mathfrak{A}$, we have $J\mathfrak{A} = \mathfrak{A}$. [Since J maps $\mathfrak{A}'' = \mathfrak{A}$ onto $\mathfrak{A}'$ we see that $\mathfrak{A} = \mathfrak{A}'$. Hence $\mathcal{R}_l(\mathfrak{A})' = \mathcal{R}_r(\mathfrak{A}') = \mathcal{R}_r(\mathfrak{A})$.]Also, it is easy to verify that for $\eta \in \mathfrak{A}$ we have

$$\begin{aligned}\pi_r(\eta) &= J\pi_l(J\eta)J, \\ \pi_r(\eta)^* &= \pi_r(\eta^\flat).\end{aligned}$$

Since $\mathfrak{A}$ is a core for $\Delta^{\frac{1}{2}}$... [What the hell does he need any of this for??]

□

3. Weights

In this section we study a generalization of positive linear functionals and the corresponding generalization of cyclic representations: semi-cyclic representations. We also characterize when such objects are "normal."

Definition 3.1. A **weight** on a von Neumanna algebras $\mathcal{M}$ is a map $\varphi\colon \mathcal{M}_+ \to [0, \infty]$ such that

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$, $x, y \in \mathcal{M}_+$;
(ii) $\varphi(\lambda x) = \lambda\varphi(x)$, $\lambda \geq 0$.

We use the convention $0 \cdot \infty = 0$. The weight is said to be **semi-finite** if

$$\mathfrak{p}_\varphi := \{x \in \mathcal{M}_+ \colon \varphi(x) < \infty\}$$

generates $\mathcal{M}$. The weight is **faithful** if $\varphi(x) \neq 0$ for every non-zero $x \in \mathcal{M}_+$. Lastly, the weight is **normal** if

$$\varphi(\sup_\alpha x_\alpha) = \sup_\alpha \varphi(x_\alpha)$$

for every bounded increasing net $\{x_\alpha\} \subset \mathcal{M}_+$.

It will rarely be the case that we consider weights which are not semi-finite (since in this case we would just replace $\mathcal{M}$ with $\mathfrak{p}_\varphi'' \cap \mathcal{M}$). Our first lemma is a more general result that is easily seen to apply to $\mathfrak{p}_\varphi$:

Lemma 3.2. *If $\mathfrak{p}$ is a herdity convex subcone of $\mathcal{M}_+$ in the sense that*

$$\begin{aligned}&\mathfrak{p} + \mathfrak{p} \subset \mathfrak{p}, \qquad \lambda\mathfrak{p} \subset \mathfrak{p}, \qquad \lambda \geq 0 \qquad (\textit{i.e. is a positive cone in } \mathcal{M}_+); \\ &0 \leq y \leq x \qquad x \in \mathfrak{p} \Longrightarrow y \in \mathfrak{p} \qquad (\textit{i.e. } y \textit{ inherits membership in } \mathfrak{p} \textit{ from } x),\end{aligned}$$

then we conclude:

(i) *$\mathfrak{n} := \{x \in \mathcal{M} \colon x^*x \in \mathfrak{p}\}$ is a left ideal of $\mathcal{M}$;*
(ii) *$\mathfrak{m} : \{\sum_{i=1}^n y_i^* x_i \colon x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{n}\}$ is a *-subalgebra such that $\mathfrak{m} \cap \mathcal{M}_+ = \mathfrak{p}$ and every element of $\mathfrak{m}$ is a linear combination of four elements of $\mathfrak{p}$.*

Proof.

(i): Since $(ax)^*ax = x^*a^*ax \leq \|a\|^2 x^*x$ we see that $a\mathfrak{n} \subset \mathfrak{n}$ for any $a \in \mathcal{M}$. Also,

$$0 \leq (x \pm y)^*(x \pm y) = x^*x \pm y^*x \pm x^*y + y^*y \Longrightarrow \mp y^*x \mp x^*y \leq x^*x + y^*y;$$

Hence

$$(x \pm y)^*(x \pm y) = x^*x \pm y^*x \pm y^*x + y^*y \leq 2(x^*x + y^*y),$$

which shows that $\mathfrak{n}$ is additive. Thus $\mathfrak{n}$ is a left ideal.

(ii): $\mathfrak{m}$ is clearly additive and closed under the * operation. Note that for $y,x,a,b\in\mathfrak{n}$, $(y^*x)(b^*a)=(x^*y)^*(b^*a)$, and since $\mathfrak{n}$ is a left ideal we know $x^*y, b^*a\in\mathfrak{n}$. Hence $\mathfrak{m}$ is closed under multiplication and therefore a *-subalgebra. The polarization identity

$$y^*x=\frac{1}{4}\sum_{k=0}^{3} i^k(x+i^ky)^*(x+i^ky),$$

easily allows us to decompose element of $\mathfrak{m}$ into a linear combination of 4 elements of $\mathfrak{p}$. Now, suppose

$$a=\sum_{i=1}^{n} y_i^*x_i\in\mathfrak{m}\cap\mathcal{M}_+,\qquad x_1,\ldots,x_n,y_1,\ldots,y_n\in\mathfrak{n}.$$

Then

$$\begin{aligned}0\leq a=\frac{1}{2}(a+a^*)=\frac{1}{2}\sum_{i=1}^{n}[y_i^*x_i+x_i^*y_i]&=\frac{1}{4}\sum_{i=1}^{n}[(x_i+y_i)^*(x_i+y_i)-(x_i-y_i)^*(x_i-y_i)]\\&\leq\frac{1}{4}\sum_{i=1}^{n}(x_k+y_k)^*(x_k+y_k)\in\mathfrak{p}.\end{aligned}$$

Since $\mathfrak{p}$ is hereditary this implies $a\in\mathfrak{p}$. Thus $\mathfrak{m}\cap\mathcal{M}_+\subset\mathfrak{p}$ and the reverse inclusion is clear. □

Definition 3.3. The sets in the above lemma defined for $\mathfrak{p}_\varphi$ are denoted:

$$\begin{aligned}\mathfrak{n}_\varphi&:=\{x\in\mathcal{M}\colon x^*x\in\mathfrak{p}_\varphi\};\\ \mathfrak{m}_\varphi&:=\left\{\sum_{i=1}^{n}y_i^*x_i\colon x_1,\ldots x_n,y_1,\ldots,y_n\in\mathfrak{n}_\varphi\right\}.\end{aligned}$$

The later set, $\mathfrak{m}_\varphi$, is the **definition domain** of the weight φ or the **definition subalgebra** of φ. We note that we can extend φ to a linear functional on $\mathfrak{m}_\varphi$.

We do a construction analogous to the GNS construction for positive linear functionals for weights. We note that the set

$$N_\varphi:=\{x\in\mathcal{M}\colon\varphi(x^*x)=0\},$$

is a left ideal of $\mathcal{M}$ contained in $\mathfrak{n}_\varphi$. We denote by η_φ the quotient map $\mathfrak{n}_\varphi\to\mathfrak{n}_\varphi/N_\varphi$. From here on the construction is identical to the GNS for positive linear functionals. We denote the completion of $\mathfrak{n}_\varphi/N_\varphi$ with respect to the norm $\|\eta_\varphi(x)\|^2=\varphi(x^*x)$ by $\mathfrak{H}_\varphi$. Also, for $a\in\mathcal{M}$ we define $\pi_\varphi(a)\in\mathcal{B}(\mathfrak{H}_\varphi)$ by $\pi_\varphi(a)\eta_\varphi(x)=\eta_\varphi(ax)$. Hence $\{\pi_\varphi,\mathfrak{H}_\varphi\}$ is a representation of $\mathcal{M}$.

Proposition 3.4. *If φ is a semi-finite normal weight, then the representation $\{\pi_\varphi,\mathfrak{H}_\varphi\}$ is a non-degenerate normal *-representation. In addition, if φ is faithful, then so is π_φ.*

Proof. That π_φ is a *-representation is clear from our experience with the GNS construction. Since $\mathcal{M}$ has a unit, $\pi_\varphi(M)\mathfrak{H}_\varphi\supset\pi_\varphi(1)\eta_\varphi(\mathfrak{n}_\varphi)=\eta_\varphi(\mathfrak{n}_\varphi)$ so the representation is non-degenerate. If φ is normal (and faithful) then it follows immediately that π_φ is normal and faithful. □

Definition 3.5. The triplet $\{\pi_\varphi,\mathfrak{H}_\varphi,\eta_\varphi\}$ is called **the semi-cyclic** representation of $\mathcal{M}$ with respect to φ. In general, a **semi-cyclic** representation of $\mathcal{M}$ is a triplet $\{\pi,\mathfrak{H},\eta\}$ consisting of a representation $\{\pi,\mathfrak{H}\}$ and a linear map $\eta\colon\mathfrak{n}\to\mathfrak{H}$ from a left ideal of $\mathcal{M}$ into $\mathfrak{H}$ with dense range such that

$$\pi(a)\eta(x)=\eta(ax),\qquad a\in\mathcal{M},\ x\in\mathfrak{n}.$$

Our first goal is to establish several characterizations of normality for a weight φ (c.f. Theorem 3.12).

Lemma 3.6. *Let $M\subset\mathcal{B}(\mathfrak{H})$ be a von Neumann algebra.*

(i) *If $x,y\in\mathcal{M}$ satisfy the inequality $y^*y\leq x^*x$, then there exists uniquely an $s\in\mathcal{M}$ such that*

$$y=sx\qquad\text{and}\qquad s[x\mathfrak{H}]^\perp=\{0\}.$$

Furthermore, $\|s\|\leq 1$.

(ii) *If $\{x_\alpha\}_{\alpha\in A}$ is a family in $\mathcal{M}$ such that $a = \sum_\alpha x_\alpha^* x_\alpha$ converges strongly, and if the operators $\{s_\alpha\}$ from part (i) are determined by $x_\alpha = s_\alpha a^{1/2}$, then the sum*

$$p = \sum_{\alpha\in A} s_\alpha^* s_\alpha$$

is strongly convergent and p is the range projection of a, the projection onto $[a\mathfrak{H}]$.

(iii) *If $\{x_\alpha\}$ is a bounded increasing net in $\mathcal{M}_+$ with $x = \sup_\alpha x_\alpha$, and if $\{s_\alpha\}$ are the operators from part (i) satisfying $x_\alpha^{1/2} = s_\alpha x^{1/2}$, then $\{s_\alpha^* s_\alpha\}$ is increasing and $p = \sup_\alpha s_\alpha^* s_\alpha$ is the range projection of x, and also $\{s_\alpha\}$ converges to p strongly.*

Proof.

(i): For each $\xi \in \mathfrak{H}$, we have

$$\|y\xi\|^2 = (y^* y\xi \mid \xi) \leq (x^* x\xi \mid \xi) = \|x\xi\|^2,$$

so that the map $s_0 \colon x\mathfrak{H} \to y\mathfrak{H}$ defined by $x\xi \mapsto y\xi$ is a well-defined linear map with $\|s_0\| \leq 1$. Thus we can extend it to a bounded operator on $[x\mathfrak{H}]$, which we continue to denote s_0. Let p be the projection onto $[x\mathfrak{H}]$ and set $s = s_0 p$, then $\|s\| \leq \|s_0\|\|p\| \leq 1$. Then $sx\xi = s_0 px\xi = s_0 x\xi = y\xi$, so that $sx = y$. Also $s[x\mathfrak{H}]^\perp = \{0\}$ since p is zero on this subspace. Since $\mathfrak{H} = [x\mathfrak{H}] \oplus [x\mathfrak{H}]^\perp$, the uniquness of s is clear. Suppose $u \in \mathcal{U}(\mathcal{M}')$ then

$$usu^* x = usxu^* = uyu^* = y,$$

and so the uniqueness of s implies $s = usu^*$. Consequently $s \in \mathcal{M}'' = \mathcal{M}$.

(ii): Let p denote the range projection of a. We note that since a is positive, $[a\mathfrak{H}]^\perp = \ker(a) = \ker(a^{1/2}) = [a^{1/2}\mathfrak{H}]^\perp$. Use part (i) to produce $s_\alpha \in \mathcal{M}$ such that $x_\alpha = s_\alpha a^{1/2}$ and $s_\alpha[a^{1/2}\mathfrak{H}]^\perp$ for each $\alpha \in A$. For $\xi \in \mathfrak{H}$, set $\eta = a^{1/2}\xi$. Let B be a finite subset of A, then

$$\begin{aligned}\left(\sum_{\alpha\in B} s_\alpha^* s_\alpha \eta \mid \eta\right) &= \sum_{\alpha\in B}\left(s_\alpha^* s_\alpha a^{\frac12}\xi \mid a^{\frac12}\xi\right) = \sum_{\alpha\in B}\left(a^{\frac12} s_\alpha^* s_\alpha a^{\frac12}\xi \mid \xi\right)\\ &= \left(\sum_{\alpha\in B} x_\alpha^* x_\alpha \xi \mid \xi\right) \leq (a\xi \mid \xi) = \left\|a^{\frac12}\xi\right\|^2 = \|\eta\|^2.\end{aligned}$$

So if we denote $p_B := \sum_{\alpha\in B} s_\alpha^* s_\alpha$, then $(p_B\eta \mid \eta) \leq (p\eta \mid \eta)$ for every η in the algebraic direct sum $a^{1/2}\mathfrak{H} + (1-p)\mathfrak{H}$ (noting that $p_B(1-p) = 0$ since $p_B[a\mathfrak{H}]^\perp = 0$ and $(1-p)\mathfrak{H} \subset [a\mathfrak{H}]^\perp$). So by continuity $p_B \leq p$ on all of $\mathfrak{H}$. Now, $\{p_B\}$ is an increasing net and hence converges strongly to some operator, say $p_0 \in \mathfrak{M}$, and $p_0 \leq p$. Letting $\eta = a^{1/2}\xi$ again we have

$$(p_0\eta \mid \eta) = \lim_B \left(\sum_{\alpha\in B} s_\alpha^* s_\alpha a^{\frac12}\xi \mid a^{\frac12}\xi\right) = \lim_B \left(\sum_{\alpha\in B} x_i^* x_i \xi \mid \xi\right) = (a\xi \mid \xi) = \left\|a^{\frac12}\xi\right\|^2 = \|\eta\|^2,$$

where we have used the strong convergence of $a = \sum_\alpha x_\alpha^* x_\alpha$. Hence $(p_0\eta \mid \eta) = \|\eta\|^2$ for every $\eta \in p\mathfrak{H} = [a^{1/2}\mathfrak{H}]$ by continuity, which means $p_0 = p$.

(iii): Let p be the range projection of x and s_α from part (i) corresponding to $x_\alpha \leq x$. If $\alpha \leq \beta$ and $\eta = x^{1/2}\xi$ then

$$(s_\alpha^* s_\alpha \eta \mid \eta) = \left(s_\alpha^* s_\alpha a^{\frac12}\xi \mid a^{\frac12}\xi\right) = (x_\alpha\xi \mid \xi) \leq (x_\beta\xi \mid \xi) = (s_\beta^* s_\beta\eta \mid \eta) \leq (x\xi \mid \xi) = \|\eta\|^2,$$

so that $\{s_\alpha^* s_\alpha\}$ is increasing and majorized by p. Letting $p_0 = \sup_\alpha s_\alpha^* s_\alpha$, the same argument as in (ii) shows $p_0 = p$.

To see that s_α converges strongly to p, first note that $\|s_\alpha\xi\|^2 = (s_\alpha^* s_\alpha\xi \mid \xi) \leq (p\xi \mid \xi)$. In particular, if $\xi \in [x\mathfrak{H}]^\perp$ then $\|s_\alpha\xi\|^2 = 0$. Thus it suffices to show $\lim_\alpha s_\alpha\eta = \eta$ for $\eta = x\xi$:

$$\begin{aligned}\lim_\alpha \|s_\alpha\eta - \eta\|^2 &= \lim_\alpha (s_\alpha\eta - \eta \mid s_\alpha\eta - \eta) = \lim_\alpha (s_\alpha^* s_\alpha\eta \mid \eta) - (s_\alpha\eta \mid \eta) - (\eta \mid s_\alpha\eta) + \|\eta\|^2\\ &= 2\|\eta\|^2 - \lim_\alpha\left[(s_\alpha x\xi \mid \eta) + (\eta \mid s_\alpha x\xi)\right] = 2\|\eta\|^2 - \lim_\alpha\left[(x_\alpha^{\frac12} x^{\frac12}\xi \mid \eta) + (\eta \mid x_\alpha^{\frac12} x^{\frac12}\xi)\right]\\ &= 2\|\eta\|^2 - \left[(x\xi \mid \eta) + (\eta \mid x\xi)\right] = 0,\end{aligned}$$

where we have used the strong convergence of the x_α. □

Lemma 3.7. *If $\mathcal{M}$ is σ-finite, then every completely additive weight φ on $\mathcal{M}$, in the sense that $\varphi\left(\sum_{\alpha\in A} x_\alpha\right) = \sum_{\alpha\in A}\varphi(x_\alpha)$, $\{x_\alpha\}\subset\mathcal{M}_+$, is normal.*

Proof. As $\mathcal{M}$ is σ-finite, it admits a faithful normal state ω. Given a bounded increasing net $\{x_\alpha\}\subset\mathcal{M}_+$ with $x=\sup_\alpha x_\alpha$, we inductively construct a sequence $\{x_n\}$ from $\{x_\alpha\}$ such that $\omega(x_n)>\omega(x)-\frac{1}{n}$ for each $n\in\mathbb{N}$. Then since $x_n\leq x$, we have $y:=\lim_n x_n\leq x$ and the convergence is σ-strong. Then

$$\omega(x)-\frac{1}{n}<\omega(x_n)\leq\omega(y)\leq\omega(x),$$

for each n so that $\omega(x)=\omega(y)$. Since ω is faithful, this implies $x=y$.

Now, setting $x_0=0$ and $y_n=x_n-x_{n-1}$ for $n=1,2,\ldots$ we have $x=\sum y_n$ and since φ is completely additive

$$\varphi(x)=\sum_{n=1}^\infty\varphi(y_n)=\lim_{n\to\infty}\varphi(x_n)\leq\lim_\alpha\varphi(x_\alpha)\leq\varphi(x).$$

Hence $\varphi(x)=\sup_\alpha\varphi(x_\alpha)$, and φ is normal. □

Lemma 3.8. *Let $\{\pi_\varphi,\ \mathfrak{H}_\varphi,\ \eta_\varphi\}$ be the semicyclic representation for a normal weight φ on $\mathcal{M}$. Then there exists a unique completely positive map θ_φ from the definition subalgebra $\mathfrak{m}_\varphi$ into $\pi_\varphi(M)'_*$ determined by the formula:*

$$\langle a,\theta_\varphi(y^*x)\rangle=(a\eta_\varphi(x)\mid\eta_\varphi(y)),\qquad a\in\pi_\varphi(M)',\ x,y\in\mathfrak{n}_\varphi.$$

Proof. Since every element of $\mathfrak{m}_\varphi$ is a sum of elements y^*x with $x,y\in\mathfrak{n}_\varphi$, it is clear that the above formula determines θ_φ by linearity.

Suppose $x^*x=y^*y$. Applying Lemma 3.6.(i) we obtain $y=sx$ and $x=ty$. The uniqueness condition implies that s is a partial isometry with $t=s^*$. Then for $a\in\pi_\varphi(M)'$, we have

$$(a\eta_\varphi(x)\mid\eta_\varphi(x))=(a\eta_\varphi(s^*y)\mid\eta_\varphi(s^*y))=(\pi_\varphi(s)a\pi_\varphi(s^*)\eta_\varphi(y)\mid\eta_\varphi(y))=(a\pi_\varphi(y)\mid\eta_\varphi(y)).$$

So the map $x^*x\in\mathfrak{m}_\varphi^+\mapsto\theta_\varphi(x^*x)\in\pi_\varphi(\mathcal{M})'_*$ is well-defined. It is clear that $\theta_\varphi(\lambda x^*x)=\lambda\theta_\varphi(x^*x)$ for $\lambda>0$.

We need to show the additivity. Let $z=y+x$ for $x,y\in\mathfrak{m}_\varphi^+$. Using Lemma 3.6.(ii) pick $s,t\in\mathcal{M}$ such that $x^{1/2}=sz^{1/2}$, $y^{1/2}=tz^{1/2}$, and $p=s^*s+t^*t$ is the range projection, $s_l(z)$, of z. Then for $a\in\pi_\varphi(M)'$ we have

$$\begin{aligned}\langle a,\theta_\varphi(z)\rangle&=\left(a\eta_\varphi\left(z^{\frac12}\right)\mid\eta_\varphi\left(z^{\frac12}\right)\right)=\left(a\pi_\varphi(s^*s+t^*t)\eta_\varphi\left(z^{\frac12}\right)\mid\eta_\varphi\left(z^{\frac12}\right)\right)\\&=\left(a\pi_\varphi(s^*s)\eta_\varphi\left(z^{\frac12}\right)\mid\eta_\varphi\left(z^{\frac12}\right)\right)+\left(a\pi_\varphi(t^*t)\eta_\varphi\left(z^{\frac12}\right)\mid\eta_\varphi\left(z^{\frac12}\right)\right)\\&=\left(a\eta_\varphi\left(x^{\frac12}\right)\mid\eta_\varphi\left(x^{\frac12}\right)\right)+\left(a\eta_\varphi\left(y^{\frac12}\right)\mid\eta_\varphi\left(y^{\frac12}\right)\right)\\&=\langle a,\theta_\varphi(x)\rangle+\langle a,\theta_\varphi(y)\rangle=\langle a,\theta_\varphi(x)+\theta_\varphi(y)\rangle,\end{aligned}$$

hence $\theta_\varphi(z)=\theta_\varphi(x)+\theta_\varphi(y)$. So is θ_φ is linear on $\mathfrak{m}_\varphi^+$ and we can then extend it to all of $\mathfrak{m}_\varphi$.

Lastly, we check it is completely positive. Let $x_1,\ldots,x_n\in\mathfrak{n}_\varphi$ and $a_1,\ldots,a_n\in\pi_\varphi(M)'$, then we have

$$\begin{aligned}\langle[a_i^*a_j],(\theta_\varphi)_n[x_i^*x_j]\rangle=\langle[a_i^*a_j],[\theta_\varphi(x_i^*x_j)]\rangle&=\sum_{i,j=1}^n\langle a_i^*a_j,\theta_\varphi(x_i^*x_j)\rangle\\&=\sum_{i,j=1}^n(a_i^*a_j\eta_\varphi(x_j)\mid\eta_\varphi(x_i))=\left\|\sum_{j=1}^n a_j\eta_\varphi(x_j)\right\|^2\geq0,\end{aligned}$$

so that θ_φ is completely positive. □

Lemma 3.9. *With the notation of the previous lemma, if $h\in\mathfrak{m}_\varphi$ is self-adjoint, then*

$$\|\theta_\varphi(h)\|=\inf\{\varphi(a)+\varphi(b)\colon h=a-b,\ a,b\in\mathfrak{m}_\varphi^+\}.$$

Proof. The $\rho(h)$ be the quantity on the right hand side of the above equation. So ρ is a function on the self-adjoint elements, $\mathfrak{m}_{\varphi,h}$, and

$$\rho(\lambda h)=|\lambda|\rho(h)\geq0\qquad\lambda\in\mathbb{R}.$$

We first show that ρ is subadditive. Fix $x_1,x_2\in\mathfrak{m}_{\varphi,h}$ and $\epsilon>0$. Let $y_i,z_i\in\mathfrak{m}_\varphi^+$ be such that $x_i=y_i-z_i$

$$\varphi(y_i)+\varphi(z_i)<\rho(x_i)+\epsilon,$$

for $i = 1, 2$. Let $x = x_1 + x_2$, $y = y_1 + y_2$, and $z = z_1 + z_2$, then $x = y - z$ and

$$\rho(x) \leq \varphi(y) + \varphi(z) \leq \rho(x_1) + \rho(x_2) + 2\epsilon.$$

Thus $\rho(x) \leq \rho(x_1) + \rho(x_2)$. Thus ρ is subadditive and is a semi-norm on $\mathfrak{m}_{\varphi,h}$. Note that ρ and φ agree on $\mathfrak{m}_\varphi^+$. For $x \in \mathfrak{m}_\varphi^+$ we have $\theta_\varphi(x) \geq 0$ and

$$\|\theta_\varphi(x)\| = \langle 1, \theta_\varphi(x)\rangle = \left\|\eta_\varphi\left(x^{\frac{1}{2}}\right)\right\|^2 = \varphi(x) = \rho(x).$$

So if $x = y - z$ with $y, z \in \mathfrak{m}_\varphi^+$ then

$$\|\theta_\varphi(x)\| \leq \|\theta_\varphi(y)\| + \|\theta_\varphi(z)\| = \varphi(y) + \varphi(z).$$

Since this holds for an arbitrary pair y, z we get $\|\theta_\varphi(x)\| \leq \rho(x)$ for $x \in \mathfrak{m}_{\varphi,h}$.

Conversely, fix $x_0 \in \mathfrak{m}_{\varphi,h}$. Using Hahn-Banach, let ψ be a real valued linear functional on $\mathfrak{m}_{\varphi,h}$ such that

$$\psi(x_0) = \rho(x_0), \qquad |\psi(x)| \leq \rho(x), \qquad x \in \mathfrak{m}_{\varphi,h}.$$

Then linearly extend ψ to all of $\mathfrak{m}_\varphi$ as a self-adjoint linear functional. We have

$$-\varphi(x^*x) = -\rho(x^*x) \leq \psi(x^*x) \leq \rho(x^*x) = \varphi(x^*x),$$

so that the sequilinear form $(\eta_\varphi(x), \eta_\varphi(y)) \mapsto \psi(y^*x)$ (defined on $\eta_\varphi(\mathfrak{n}_\varphi) \times \eta_\varphi(\mathfrak{n}_\varphi)$) is bounded and hence extends to all of $\mathfrak{H}_\varphi$. As it is bounded, there is a bounded operator $a \in \mathcal{B}(\mathfrak{H}_\varphi)$ with $\|a\| \leq 1$ such that

$$(a\eta_\varphi(x) \mid \eta_\varphi(y)) = \psi(y^*x), \qquad x, y \in \mathfrak{n}_\varphi.$$

It is easy to see that $a \in \pi_\varphi(M)'$. Since ψ is self-adjoint, a is also self-adjoint. As $x_0 = \sum_i y_i^* x_i$ is self-adjoint we have

$$x_0 = \frac{1}{2}(x_0 + x_0^*) = \frac{1}{2}\sum_i y_i^* x_i + x_i^* y_i = \frac{1}{4}\sum_i (x_i + y_i)^*(x_i + y_i) - (x_i - y_i)^*(x_i - y_i) = \frac{1}{4}\sum_i p_i^* p_i - q_i^* q_i,$$

where $p_i = x_i + y_i \in \mathfrak{n}_\varphi$ and $q_i = x_i - y_i \in \mathfrak{n}_\varphi$. Then

$$\begin{aligned}\rho(x_0) = \psi(x_0) &= \frac{1}{4}\sum_i \psi\left(p_i^* p_i\right) - \psi(q_i^* q_i) = \frac{1}{4}\sum_i \left(a\eta_\varphi\left(p_i\right) \mid \eta_\varphi\left(p_i\right)\right) - \left(a\eta_\varphi(q_i) \mid \eta_\varphi(q_i)\right) \\ &= \frac{1}{4}\sum_i \langle a, \theta_\varphi(p_i^* p_i)\rangle - \langle a, \theta_\varphi(q_i^* q_i)\rangle = \left\langle a, \theta_\varphi\left(\frac{1}{4}\sum_i p_i^* p_i - q_i^* q_i\right)\right\rangle = \langle a, \theta_\varphi(x_0)\rangle \leq \|\theta_\varphi(x_0)\|,\end{aligned}$$

Thus $\rho(x_0) \leq \|\theta_\varphi(x_0)\|$ and we obtain equality. □

Our next lemma asserts that θ_φ is closed when restricted to $\mathfrak{m}_\varphi^+$.

Lemma 3.10. *We again maintain the same notation of the previous lemmas. Suppose $\{x_n\}$ is a bounded sequence of $\mathfrak{m}_\varphi^+$.*

(i) *If $\{x_n\}$ converges to $x \in \mathcal{M}$ σ-strongly and if $\{\theta_\varphi(x_n)\}$ converges in norm, then x belongs to $\mathfrak{m}_\varphi^+$.*
(ii) *If $\{x_n\}$ converges to zero σ-strongly and if $\{\theta_\varphi(x_n)\}$ converges in norm, then the limit of $\{\theta_\varphi(x_n)\}$ must be zero.*

Proof.

(i): Fix $\epsilon > 0$ and let $\psi = \lim_{n\to\infty} \theta_\varphi(x_n)$. Choose a subsequnence $\{y_n\} \subset \{x_n\}$ so that $\|\psi - \theta_\varphi(y_n)\| < \epsilon/2^{n+1}$. This implies

$$\|\theta_\varphi(y_{n+1}) - \theta_\varphi(y_n)\| < \frac{\epsilon}{2^n}.$$

By the previous lemma, find $a_n, b_n \in \mathfrak{m}_\varphi^+$ with $y_{n+1} - y_n = a_n - b_n$ and

$$\varphi(a_n) + \varphi(b_n) < \frac{\epsilon}{2^n}.$$

Then

$$y_{n+1} = y_1 + \sum_{k=1}^n y_{k+1} - y_k = y_1 + \sum_{k=1}^n a_k - b_k \leq y_1 + \sum_{k=1}^n a_k.$$

For $\alpha > 0$, we define a function f_α on $(-1/\alpha, \infty)$ by

$$f_\alpha(t) := \frac{t}{1+\alpha t}.$$

Then f_α is operator monotone: $-1/\alpha < x \leq y \Longrightarrow f_\alpha(x) \leq f_\alpha(y)$. Also $f_\alpha(t) \leq \frac{1}{\alpha}$. Applying this to the above inequality we obtain

$$f_\alpha(y_{n+1}) \leq f_\alpha\left(y_1 + \sum_{k=1}^n a_k\right) \leq \frac{1}{\alpha}.$$

Hence $\{f_\alpha(y_1 + \sum_{k=1}^n a_k)\}$ is bounded and increasing and therefore converges σ-strongly to some $c_\alpha \in \mathcal{M}_+$. Then

$$c_\alpha = \lim_{n\to\infty} f_\alpha\left(y_1 + \sum_{k=1}^n a_k\right) \geq \lim_{n\to\infty} f_\alpha(y_{n+1}) = f_\alpha(x),$$

where we have used the fact that the functional calculus for bounded functions on closed subsets of $\mathbb{C}$ (restrict $f_\alpha|_{[0,\infty)}$) is strongly continuous (see Lemma II.4.6 in [2]). Next, using $f_\alpha(t) \leq t$ and the normality of φ we get

$$\begin{aligned}\varphi(f_\alpha(x)) &\leq \lim_{n\to\infty} \varphi\left(f_\alpha\left(y_1 + \sum_{k=1}^n a_k\right)\right) \leq \lim_{n\to\infty} \varphi\left(y_1 + \sum_{k=1}^n a_k\right)\\ &= \varphi(y_1) + \sum_{k=1}^\infty \varphi(a_k) \leq \varphi(y_1) + \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \varphi(y_1) + \epsilon.\end{aligned}$$

Now since $t = \lim_{\alpha\to 0} f_\alpha(t)$ and $f_\alpha(t) \leq f_\beta(t)$ if $0 < \beta < \alpha$, the normality of φ yields

$$\varphi(x) = \lim_{\alpha\to 0} \varphi(f_\alpha(x)) \leq \varphi_1(y_1) + \epsilon < +\infty,$$

hence $x \in \mathfrak{p}_\varphi = \mathfrak{m}_\varphi^+$.

(ii) Fix $\epsilon > 0$, ψ, $\{y_n\}$, $\{a_n\}$, $\{b_n\}$, and f_α as before. We note

$$y_1 - y_{n+1} = \sum_{k=1}^n y_k - y_{k+1} = \sum_{k=1}^n b_k - a_k \leq \sum_{k=1}^n b_k.$$

The uniform boundedness theorem implies $\{y_n\}$ is bounded, say $K = \sup \|y_n\|$. Then $y_1 - y_{n+1} \geq -K$, so that if $0 < \alpha < 1/K$, we can apply f_α to obtain

$$f_\alpha(y_1 - y_{n+1}) \leq f_\alpha\left(\sum_{k=1}^n b_k\right).$$

Let $d_\alpha = \lim_{n\to\infty} f_\alpha(\sum_{k=1}^n b_k)$. By assumption, $\lim_{n\to\infty} y_n = 0$, so

$$f_\alpha(y_1) = \lim_{n\to\infty} f_\alpha(y_1 - y_{n+1}) \leq \lim_{n\to\infty} f_\alpha\left(\sum_{k=1}^n b_k\right) = d_\alpha.$$

Normality gives us

$$\varphi(f_\varphi(y_1)) \leq \varphi(d_\alpha) = \lim_{n\to\infty} \varphi\left(f_\alpha\left(\sum_{k=1}^n b_k\right)\right),$$

and

$$\varphi\left(f_\alpha\left(\sum_{k=1}^n b_k\right)\right) \leq \varphi\left(\sum_{k=1}^n b_k\right) < \sum_{k=1}^n \frac{\epsilon}{2^k} < \epsilon$$

implies that $\varphi(f_\alpha(y_1)) \leq \epsilon$. As $\alpha \to 0$, $f_\alpha(y_1)$ converges upwards to y_1 and we get

$$\varphi(y_1) = \lim_{\alpha\to 0} \varphi(f_\alpha(y_1)) \leq \epsilon.$$

Hence

$$\|\psi\| \leq \|\psi - \theta_\varphi(y_1)\| + \|\theta_\varphi(y_1)\| < \epsilon + \varphi(y_1) < 2\epsilon,$$

so it follows that $\psi = 0$. $\square$

Definition 3.11. A weight φ is σ-weakly lower semi-continuous if for each $t > 0$ the set

$$\{x \in \mathcal{M}_+ \colon \varphi(x) \leq t\}$$

is σ-weakly closed.

Theorem 3.12. *If φ is a weight on a von Neumann algebra $\mathcal{M}$, then the following are equivalent:*

(i) *φ is completely additive in the sense that*

$$\varphi\left(\sum_{\alpha \in A} x_\alpha\right) = \sum_{\alpha \in A} \varphi(x_\alpha)$$

for every σ-strongly summable family $\{x_\alpha\}$ in $\mathcal{M}_+$;

(ii) *φ is normal;*

(iii) *φ is σ-weakly lower semi-continuous;*

(iv) *If we define*

$$\Phi_\varphi := \{\omega \in \mathcal{M}_*^+ \colon \omega(x) \leq \varphi(x),\ x \in \mathcal{M}_+\},$$

then

$$\varphi(x) = \sup\{\omega(x) \colon \omega \in \Phi_\varphi\}, \qquad x \in \mathcal{M}_+$$

We aren't quite ready to prove fully prove this, but we note that $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ are easy [ADD PROOF]. The difficult implication is $(i) \to (iv)$.

Consider the graph of η_φ:

$$G := \{(x, \eta_\varphi(x)) \in \mathcal{M} \oplus \mathfrak{H}_\varphi \colon x \in \mathfrak{n}_\varphi\}.$$

On the direct sum $\mathcal{M} \oplus \mathfrak{H}_\varphi$ we take the norm:

$$\|(x, \xi)\| := \max\{\|x\|, \|\xi\|\}, \qquad x \in \mathcal{M},\ \xi \in \mathfrak{H}_\varphi.$$

Hence it is the Banach space dual of the direct sum $\mathcal{M}_* \oplus \mathfrak{H}_\varphi^*$ with norm:

$$\|(\omega, \xi^*)\| := \|\omega\| + \|\xi^*\|, \qquad \omega \in \mathcal{M}_*,\ \xi^* \in \mathfrak{H}_\varphi^*.$$

Lemma 3.13. *If $\mathcal{M}$ is σ-finite, then the unit ball of G is weak* compact in the above Banach space $\mathcal{M} \oplus \mathfrak{H}_\varphi$.*

Proof. Let B be the closed unit ball of $\mathcal{M} \oplus \mathfrak{H}_\varphi$, then by Alaoglu's theorem it is weak* compact and so we only need to show $B \cap G$ is weak* closed. Since $B \cap G$ is convex, the Krein-Smulian Theorem (see Lemma 12.1 in [1]) implies that suffices to show $B \cap G$ is closed under a locally convex topology on $\mathcal{M} \oplus \mathfrak{H}_\varphi$ having $\mathcal{M}_* \oplus \mathfrak{H}_\varphi^*$ as the dual space. Towards this end, we consider the product topology $\mathcal{T}$ of the σ-strong* topology on $\mathcal{M}$ and the norm topology on $\mathfrak{H}_\varphi$, and prove that $B \cap G$ is $\mathcal{T}$-closed.

As $\mathcal{M}$ is σ-finite, the unit ball is metrizable with respect to the σ-strong* topology. Therefore, if (x, ξ) is a limit point of $B \cap G$, then there exists a sequence $\{x_n\}$ in $\mathfrak{n}_\varphi$ such that $\{x_n\}$ converges to x σ-strongly* and $\|\eta_\varphi(x_n) - \xi\| \to 0$. Since $(x_n, \eta_\varphi(x_n)) \in B$, we have $\|x_n\| \leq 1$ and $\|\eta_\varphi(x_n)\| \leq 1$, so that $\{x_n^* x_n\}$ converges to $x^* x$ σ-strongly:

$$\begin{aligned}
\sum_{m=1}^{\infty} \|(x_n^* x_n - x^* x)\xi_m\|^2 &\leq 2 \sum_{m=1}^{\infty} \|(x_n^* x_n - x_n^* x)\xi_m\|^2 + \|(x_n^* x - x^* x)\xi_m\|^2 \\
&\leq 2 \sum_{m=1}^{\infty} \|x_n^*\|^2 \|(x_n - x)\xi_m\|^2 + \|(x_n^* - x^*)(x\xi_m)\|^2 \\
&\leq 2 \sum_{m=1}^{\infty} \|(x_n - x)\xi_m\|^2 + \|(x_n^* - x^*)\eta_n\|^2,
\end{aligned}$$

where $\sum_m \|\xi_m\|^2 < \infty$ and $\sum_m \|\eta_m\|^2 \leq \|x\|^2 \sum_m \|\xi_m\|^2 < \infty$. We also have that $\theta_\varphi(x_n^* x_n) = \omega_{\eta_\varphi(x_n)}$ converges to ω_ξ in norm, where ω_ζ, $\zeta \in \mathfrak{H}_\varphi$, means the vectorial functional: $a \in \pi_\varphi(\mathcal{M})' \to (a\zeta \mid \zeta)$. Indeed, given $\epsilon > 0$ let n be such that $\|\eta_\varphi(x_n) - \xi\| < \min\left\{\frac{\epsilon}{2}, \frac{\epsilon}{2\|\xi\|}\right\}$. Then let $a \in \pi_\varphi(M)'$ with $\|a\| \leq 1$. We have

$$\begin{aligned}
\|\omega_{\eta_\varphi(x_n)}(a) - \omega_\xi(a)\| &\leq \|(a\eta_\varphi(x_n) \mid \eta_\varphi(x_n)) - (a\xi \mid \eta_\varphi(x_n)\| + \|(a\xi \mid \eta_\varphi(x_n)) - (a\xi \mid \xi)\| \\
&\leq \|(a(\eta_\varphi(x_n) - \xi) \mid \eta_\varphi(x_n))\| + \|(a\xi \mid \eta_\varphi(x_n) - \xi)\| \\
&\leq \|a\| \|\eta_\varphi(x_n) - \xi\| \|\eta_\varphi(x_n)\| + \|a\| \|\xi\| \|\eta_\varphi(x_n) - \xi\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus we can apply Lemma 3.10.(i) to obtain that $x^*x \in \mathfrak{m}_\varphi^+$, ergo $x \in \mathfrak{n}_\varphi$. Thus we can define $\theta_\varphi((x_n - x)^*(x_n - x)) = \omega_{\eta_\varphi(x_n - x)}$. We know $\{(x_n - x)^*(x_n - x)\}$ converges to zero σ-strongly and by the same argument as above, $\omega_{\eta_\varphi(x_n - x)}$ converges to $\omega_{\xi - \eta_\varphi(x_n)}$ in norm. Hence Lemma 3.10.(ii) implies that $\omega_{\xi-\eta_\varphi(x)} = 0$ inn $\pi_\varphi(\mathcal{M})'_*$. In particular, since this functional is positive we see that

$$\|\xi - \eta_\varphi(x)\|^2 = \omega_{\xi - \eta_\varphi(x)}(1) = 0.$$

Thus $\xi = \eta_\varphi(x)$, and $(x, \xi) \in B \cap G$. As (x, ξ) was an arbitrary limit point, $B \cap G$ is $\mathcal{T}$-closed. □

Lemma 3.14. *If $\mathcal{M}$ is σ-finite, then conditions (i), (ii), and (iii) in Theorem 3.12 are equivalent.*

Proof. $(ii) \Rightarrow (i)$ is clear and the other direction was show in Lemma 3.7. Also, $(iii) \Rightarrow (i)$ is clear from the definition. We show $(ii) \Rightarrow (iii)$.

For each $r, s > 0$ we set

$$B_{r,s} := \{(x, \xi) \in \mathcal{M} \oplus \mathfrak{H}_\varphi \colon \|x\| \leq r,\ \|\xi\| \leq s\}.$$

Then by the previous lemma, $B_{r,s} \cap G$ is weak* compact, so that the projection $C_{r,s}$ of $B_{r,s} \cap G$ in $\mathcal{M}$:

$$C_{r,s} = \{x \in \mathcal{M} \colon \|x\| \leq r,\ \varphi(x^*x) \leq s^2\}$$

is σ-weakly compact. Set

$$E_s := \{x \in \mathcal{M} \colon \varphi(x^*x) \leq s^2\}, \qquad s > 0.$$

Let S be the closed unit ball of $\mathcal{M}$. Note that E_r is convex:

$$\begin{aligned}\varphi((tx + (1-t)y)^*(tx + (1-t)y))^{1/2} &= \|\eta_\varphi(tx + (1-t)y)\| \\ &\leq t\|\eta_\varphi(x)\| + (1-t)\|\eta_\varphi(y)\| \leq ts + (1-t)s = s.\end{aligned}$$

Hence the σ-weak compactness of $E_s \cap rS = C_{r,s}$ for every $r > 0$ implies that E_s is σ-weakly closed. Next, set

$$F_s := \{x \in \mathcal{M}_+ \colon \varphi(x) \leq s^2\}$$

and note that by definition φ is σ-weakly lower semi-continuous iff F_s is σ-weakly closed for each s. So it suffices to show the latter. Convexity of F_s (shown with the same reasoning as with E_s) implies it is enough to show that $F_s \cap rS$ is σ-strongly closed. Suppose $\{x_\alpha\}$ is a net in $F_s \cap rS$ converging σ-strongly to some $x \in \mathcal{M}_+$. As the square root operation is σ-strongly continuous on $F_s \cap rS$, we know $\{x_\alpha^{1/2}\}$ converges σ-strongly to $x^{1/2}$. But $\{x_\alpha^{1/2}\} \subset E_s$, which is σ-weakly closed. Hence $x^{1/2} \in E_s$ and so $x \in F_s$. □

We require some lemmas to extend this result from the σ-finite case to the general case.

Lemma 3.15. *Let Σ denote the set of σ-finite projections of $\mathcal{M}$ and $\mathcal{M}_0 := \bigcup\{p\mathcal{M}p \colon p \in \Sigma\}$. Then $\mathcal{M}_0$ is an ideal of $\mathcal{M}$ and the limit of each σ-weakly convergent sequence belongs to $\mathcal{M}_0$.*

Proof. We first note that $x \in M_0$ iff $s_l(x), s_r(x) \in \Sigma$ (where $s_l(x)$ and $s_r(x)$ are the range projections of x and x^* respectively). Indeed, suppose $x \in \mathcal{M}_0$, then $x \in p\mathcal{M}p$ for some $p \in \Sigma$. Hence $x = px$ so that $s_l(x) \leq p$ and thus $s_l(x) \in \Sigma$. Also $x = xp$, so that $x^* = px^*$ and consequently $s_r(x) \leq p$ and $s_r(x) \in \Sigma$. Conversely, if $s_l(x), s_r(x) \in \Sigma$ then $p : s_l(x) \wedge s_r(x) \in \Sigma$ and $x = pxp$. It is clear that $\mathcal{M}_0$ is additive, but with this characterization we can see it is in fact an ideal: for $x \in \mathcal{M}_0$ and $a \in \mathcal{M}$ we have $s_r(ax) = s_l(x^*a^*) \leq s_l(x^*)$ and so $s_r(ax) \in \Sigma$. But also, $s_l(ax) \in \Sigma$ since it is equivalent to $s_r(ax)$ (by considering the polar decomposition).

Let $\{x_n\} \subset \mathcal{M}_0$ be a sequence converging to x σ-weakly. Then there exists a sequence $\{p_n\} \subset \Sigma$ such that $x_n = p_n x_n p_n$ for each n. Let $p = \bigwedge_{n=1}^\infty p_n$, then $p \in \Sigma$ and $x_n = px_np$ for each n. But then $x = pxp$ and hence $x \in \mathcal{M}_0$. □

Lemma 3.16. *Letting the notation be as in the above lemmas, suppose that a convex set $F \subset \mathcal{M}_0^+$ is hereditary. A necessary and sufficient condition for F to be σ-weakly closed in $\mathcal{M}_0$ is that $F \cap p\mathcal{M}p$ is σ-weakly closed for every $p \in \Sigma$.*

Proof. The necessity of the condition is clear, so we show the sufficiency: suppose $F \cap p\mathcal{M}p$ is σ-weakly closed for every $p \in \Sigma$. Let ω be a normal state on $\mathcal{M}$ with $p = s(\omega)$, the support of ω. Define $d(x, y) :=$

$\omega((x-y)^*(x-y))^{1/2}$ for $x, y \in p\mathcal{M}p \cap S$ (where S is the closed unit ball of $\mathcal{M}$), then it is a metric. Set $E = \{x \in \mathcal{M} \colon x^*x \in F\}$. Then for $x, y \in E$ and $0 \leq \lambda \leq 1$ we have

$$\begin{aligned} 0 &\leq (\lambda x + (1-\lambda)y)^*(\lambda x + (1-\lambda)y) = \lambda^2 x^*x + (1-\lambda)^2 y^*y + \lambda(1-\lambda)(x^*y + y^*x) \\ &\leq \lambda^2 x^*x + (1-\lambda)^2 y^*y + \lambda(1-\lambda)(x^*x + y^*y) = \lambda x^*x + (1-\lambda)y^*y \in F. \end{aligned}$$

Thus $\lambda x + (1-\lambda)y \in E$ as F is hereditary. Thus E is convex. Also, if $a \in S$, then $aE \subset E$ since $(ax)^*ax \leq x^*x$.

Next we show that $E \cap p\mathcal{M}p$ is σ-weakly closed for every $p \in \Sigma$. Indeed, by the convexity of the set it suffices to show $E \cap p\mathcal{M}p \cap rS$ is σ-strong* closed for $r > 0$. But this set is simply the inverse image of $F \cap p\mathcal{M}p$ under the map: $x \mapsto x^*x$ from $rS \cap p\mathcal{M}p \to r^2 S \cap p\mathcal{M}p$. Since this map is σ-strong* continuous and $F \cap p\mathcal{M}p$ is σ-weakly closed we are done.

Then we show pE is σ-weakly closed for all $p \in \Sigma$, which is equivalent to that of E^*p. It again suffices to show that $rS \cap E^*p$ is σ-strong closed for $r > 0$. If x is in the σ-strong closure of $rS \cap E^*p$, then x is approximated σ-strongly by $\{x_n^*\} \subset rS \cap E^*p$. Thus, $\{x_n^*\}$ converges σ-weakly to x^*. Since $\mathcal{M}_0$ is an ideal and $p = ppp \in \mathcal{M}_0$, we get that $x_n^* \in \mathcal{M}_0$ for each n. Let $q \in \Sigma$ be such that $x_n = qx_nq$ for every n. Thus $x_n \in E^*p \cap q\mathcal{M}q \subset E^* \cap q\mathcal{M}q$ (since $p \in S$ and for $a \in S$ we have $aE \subset E$). The set $E^* \cap q\mathcal{M}q$ is σ-weakly closed by the previous argument (which showed $E \cap q\mathcal{M}q$ is σ-weakly closed), hence we have $x^* \in E^* \cap q\mathcal{M}q$. So we have $x^* \in E^*$. But also, $x^*p = \lim_n x_n^*p = \lim_n x_n^* = x^*$, so that $x^* \in \mathcal{M}p$. Thus $x^* \in E^* \cap \mathcal{M}p = E^*p$, and $x^* \in rS$ is clear. So we have the required σ-strong closedness.

Let $\tilde{F}$ be the σ-strong closure of F and $y \in \tilde{F} \cap \mathcal{M}_0$. Set $p := s_l(y) \in \Sigma$ and let $\{y_i\}$ be a net in F converging σ-strongly to y. Then $\{y_i^{1/2}\}$ converges to $y^{1/2}$ σ-strongly, and thus $\{py_i^{1/2}\}$ converges to $py^{1/2}$. The closedness of pE implies that $py^{1/2} = y^{1/2} \in pE \subset E$. Therefore, y belongs to F, and $\tilde{F} \cap \mathcal{M}_0 = F$. □

Proof of $(i) \Rightarrow (iii)$ in Theorem 3.12. Suppose φ is a completely additive weight on $\mathcal{M}$ in the sense of (i). By Lemma 3.14, φ is a σ-weakly lower semi-continuous on $(p\mathcal{M}p)_+$ for each $p \in \Sigma$. Set $F := \{x \in \mathcal{M}_+ \colon \varphi(x) \leq 1\}$, then $F \cap p\mathcal{M}p$ is a σ-weakly closed. Also, F is convex and hereditary, so the previous lemma implies $F \cap \mathcal{M}_0$ is relatively σ-weakly closed in $\mathcal{M}_0$. Let $\{p_i\}_{i \in I}$ be a maximal orthogonal family on Σ. Then $\sum_i p_i = 1$. For each finite subset $J \subset I$, set $q_J := \sum_{i \in J} p_i$. It follows that $q_J \in \Sigma$ and $\{q_J\}$ increases up to 1. To show the σ-weak closedness of F, it suffices to prove the σ-strong closedness of $F \cap rS$, $r > 0$, by the convexity of F. Let $\{x_\lambda \colon \lambda \in \Lambda\}$ be a net in $F \cap rS$ converging σ-strongly to x. For each finite subset J of I, $\{x_\lambda^{1/2} q_J x_\lambda^{1/2} \colon \lambda \in \Lambda\}$ converges to $x^{1/2} q_J x^{1/2}$ σ-strongly. Since $\mathcal{M}_0$ is an ideal of $\mathcal{M}$, both $\{x_\lambda^{1/2} q_J x_\lambda^{1/2}\}$ and $x^{1/2} q_J x^{1/2}$ belong to $\mathcal{M}_0$. Since $x_\lambda^{1/2} q_J x_\lambda^{1/2} \in F$, $x^{1/2} q_J x^{1/2} \in F \cap \mathcal{M}_0$ as seen above. Namely, we have $\varphi(x^{1/2} q_J x^{1/2}) \leq 1$, thus we conclude, using the complete additivity of φ:

$$\varphi(x) = \varphi\left(\sum_{i \in I} x^{\frac{1}{2}} p_i x^{\frac{1}{2}}\right) = \sum_{i \in I} \varphi\left(x^{\frac{1}{2}} p_i x^{\frac{1}{2}}\right) = \lim_J \sum_{i \in J} \varphi\left(x^{\frac{1}{2}} p_i x^{\frac{1}{2}}\right) = \lim_J \varphi\left(x^{\frac{1}{2}} q_J x^{\frac{1}{2}}\right) \leq 1.$$

Hence $x \in F$, and $x \in rS$ is clear. Therefore, F is σ-weakly closed. Now, for each $s > 0$, we have

$$F_s := \{x \in \mathcal{M}_+ \colon \varphi(x) \leq s^2\} = s^2 F,$$

concluding the σ-weak lower semi-continuity of φ. □

Recall that we noted the ascending implications in Theorem 3.12 were clear, so the above proof establishes the equivalence of $(i), (ii)$, and (iii). It remains to show that together they imply (iv). We'll need some results about ordered locally convex vector spaces.

If A is an ordered locally convex vector space over $\mathbb{R}$, let A_+ be the positive part and assume $A = A_+ - A_+$. In the dual space, A^*, the dual positive cone is defined as:

$$A_+^* := \{\omega \in A^* \colon \omega(x) \geq 0,\ x \in A_+\}.$$

A_+^* gives A^* an ordered structure. Given a subset $F \subset A$ we define the **polar** of F by:

$$F^\circ := \{\omega \in A^* \colon \omega(x) \leq 1,\ x \in F\}.$$

Also, we denote

$$F^\wedge := F^\circ \cap A_+^* := \{\omega \in A_+^* \colon \omega(x) \leq 1,\ x \in F\}.$$

Given $E \subset A^*$, E° and $E^\wedge$ are defined analogously.

Lemma 3.17. *Let A be an ordered locally convex vector space, then the following are equivalent:*

(i) *For every hereditary convex closed subset* $F \subset A_+$

$$F = (F - A_+)^- \cap A_+,$$

where the bar denotes the closure.

(ii) *For every hereditary convex closed subset* $F \subset A_+$

$$F = F^{\wedge\wedge}.$$

(iii) *If* φ *is an extended real valued lower semi-continuous function on* A_+ *such that*

$$\left.\begin{array}{ll} \varphi(x) \leq \varphi(y), & \text{if } 0 \leq x \leq y; \\ \varphi(x+y) \leq \varphi(x) + \varphi(y), & \text{if } x, y \in A_+; \\ \varphi(\lambda x) = \lambda\varphi(x), & \text{if } \lambda \in \mathbb{R}_+, \end{array}\right\} \tag{4}$$

then φ *is of the form*

$$\varphi(x) = \sup\{\omega(x) \colon \omega \in \Phi\}, \qquad x \in A_+,$$

where

$$\Phi := \{\omega \in A_+^* \colon \omega(x) \leq \varphi(x), x \in A_+\}. \tag{5}$$

Proof.

(i)$\Rightarrow$(ii): We show $(F - A_+)^\circ = (F \cup (-A_+))^\circ$. Noting that 0 is an element of both F and $-A_+$, it is easy to see that for $\omega \in (F - A_+)^\circ$ we have $\omega(x), \omega(y) \leq 1$ for any $x \in F$ and $y \in -A_+$. Hence $(F - A_+)^\circ \subset (F \cup (-A_+))^\circ$. Conversely, let $\omega \in (F \cup (-A_+))^\circ$. Given $x \in F$ and $y \in -A_+$ we must show $\omega(x+y) \leq 1$. Let $0 < \lambda < 1$, then

$$\begin{aligned} \omega(x+y) &= \frac{1}{\lambda}\omega\left(\lambda x + \lambda y\right) = \frac{1}{\lambda}\omega\left(\lambda x + (1-\lambda)\frac{\lambda}{1-\lambda}y\right) \\ &= \frac{1}{\lambda}\left[\lambda\omega(x) + (1-\lambda)\omega\left(\frac{\lambda}{1-\lambda}y\right)\right] \leq \frac{1}{\lambda}[\lambda + (1-\lambda)] = \frac{1}{\lambda}, \end{aligned}$$

where we have used the fact that $-A_+$ is a cone to say $\frac{\lambda}{1-\lambda}y \in -A_+$. Since this holds for each $\lambda \in [0,1]$, let λ tend to 1 to obtain $\omega(x+y) \leq 1$ or $\omega \in (F - A_+)^\circ$. Hence the polars are equal. Then we have

$$(F - A_+)^\circ = (F \cup (-A_+))^\circ = F^\circ \cap (-A_+)^\circ,$$

but $(-A_+)^\circ = A_+^*$. Indeed, if $\omega \in (-A_+)^\circ$ then $\omega(x) = -\omega(-x) \geq -1$ for all $x \in A_+$. Scaling x we see $\omega(x) = \lambda\omega\left(\frac{1}{\lambda}x\right) \geq -\lambda$ for all $\lambda > 0$ and hence $\omega(x) \geq 0$, so $\omega \in A_+^*$. Conversely, if $\omega \in A_+^*$ then $\omega(-x) = -\omega(x) \leq 0 < 1$ for all $x \in A_+$ so that $\omega \in (-A_+)^\circ$. Thus we have

$$(F - A_+)^\circ = F^\circ \cap A_+^* = F^\wedge.$$

Now, the Hanhn-Banach Seperation theorem implies $(F - A_+)^- = (F - A_+)^{\circ\circ}$. Indeed, we clearly have $F - A_+ \subset (F - A_+)^{\circ\circ}$ and since $\omega(x) \leq 1$ is a closed condition the closure is contained as well. Conversely, we cannot have $x \in (F - A_+)^{\circ\circ} \setminus (F - A_+)^-$ because $(F - A_+)^-$ is a closed convex set and thus we can then find $\psi \in A^*$ such that $\psi(x) < s < t < \psi(y)$ for some $s, t \in \mathbb{R}$ and all $y \in (F - A_+)^-$. Since $-A_+ = 0 - A_+ \subset F - A_+$, this implies $s < t \leq 0$. Indeed, for any $\lambda > 0$ and $y \in -A_+$ we have $\lambda y \in -A_+ \subset (F - A_+)^-$ so that $t < \psi(\lambda y) = \lambda\psi(y)$ or

$$\frac{t}{\lambda} < \psi(y).$$

If $t > 0$, then letting $\lambda = \frac{t}{\psi(y)}$ would contradict the above inequality. Thus $s < t \leq 0$, so if $\omega := \frac{\psi}{s}$ then we have $\omega(y) = \frac{\psi(y)}{s} \leq 1$ for all $y \in F - A_+$ so that $\omega \in (F - A_+)^\circ$. On the other hand, $\omega(x) = \frac{\psi(x)}{s} > 1$, which contradicts $x \in (F - A_+)^{\circ\circ}$. Thus $(F - A_+)^- = (F - A_+)^{\circ\circ}$ and so by the above work and assumption we obtain

$$F = (F - A_+)^- \cap A_+ = (F - A_+)^{\circ\circ} \cap A_+ = (F^\wedge)^\circ \cap A_+ = F^{\wedge\wedge}$$

(ii)⇒(iii): Suppose φ is a function on A_+ satisfying the conditions (4) in (iii) and set $F = \{x \in A_+ : \varphi(x) \leq 1\}$. Then F is a hereditary closed convex set. Realize that Φ in equation (5) is exactly $F^\wedge$. Indeed, if $\omega \in \Phi$ then $\omega(x) \leq \varphi(x)$ for all $x \in A_+$ and $\omega \in A_+^*$. In particular, $\omega(x) \leq \varphi(x) \leq 1$ for all $x \in F$, hence $\omega \in F^\wedge$. Conversely, if $\omega \in F^\wedge$ then for $x \in A_+$ we either have $\varphi(x) = 0$ or $\varphi(x) \neq 0$. In the former case, we have $\varphi(\lambda x) = \lambda\varphi(x) = 0$ for all $\lambda > 0$ so that $\lambda x \in F$ for all λ. Hence $\omega(x) = \frac{1}{\lambda}\omega(\lambda x) \leq \frac{1}{\lambda}$ for all λ. Letting λ tend to infinity we have $\omega(x) = 0$ and thus $\omega(x) \leq \psi(x)$ holds. If $\varphi(x) \neq 0$ then let $y = \frac{x}{\varphi(x)}$, so that $\varphi(y) = 1$ and hence $y \in F$. But then $1 \geq \omega(y) = \frac{1}{\varphi(x)}\omega(x)$ or $\omega(x) \leq \varphi(x)$. So $F^\wedge = \Phi$. Set

$$\psi(x) := \sup\{\omega(x) : \omega \in \Phi\}, \qquad x \in A_+.$$

By definition of Φ we have $\psi(x) \leq \varphi(x)$ for all $x \in A_+$. Suppose there is some $x_0 \in A_+$ with $\psi(x_0) < \varphi(x_0)$. Scaling x_0 we may assume $\psi(x_0) < 1 < \varphi(x_0)$, in which case $x_0 \notin F$. On the other hand, (ii) implies that

$$F = F^{\wedge\wedge} = \{x \in A_+ : \omega(x) \leq 1,\ \omega \in \Phi\} = \{x \in A_+ : \psi(x) \leq 1\},$$

so that $x_0 \in F$, a contradiction. Hence $\psi = \varphi$.

(iii)⇒(i): Suppose F is a herditary convex closed subset of A_+. Set

$$\varphi(x) := \inf\{r > 0 : x \in rF\},$$

then φ satisfies the hypothesis of (iii). Since F is closed, we have $F = \{x \in A_+ : \varphi(x) \leq 1\}$. Hence

$$F = \{x \in A_+ : \omega(x) \leq 1,\ \omega \in \Phi\} = \Phi^\circ \cap A_+.$$

On the other hand

$$\Phi = \{\omega \in A_+^* : \omega(x) \leq \varphi(x),\ x \in A_+\} = \{\omega \in A_+^* : \omega(x) \leq 1, x \in F\} = F^\circ \cap A_+^*.$$

As we saw earlier, $F^\circ \cap A_+^* = (F \cup (-A_+))^\circ = (F - A_+)^\circ$. Thus $\Phi^\circ = (F - A_+)^{\circ\circ} = (F - A_+)^-$ by the separation argument. Hence $F = \Phi^\circ \cap A_+ = (F - A_+)^- \cap A_+$, as desired. □

We can now finish the proof of Theorem 3.12.

Proof of (iii) ⇒ (iv) in Theorem 3.12. It suffices to show that condition (i) in the previous lemma is satisfied for $A = \mathcal{M}_h$. Note that the relevant topology is the σ-strong topology.

Let F be a σ-strongly closed hereditary convex subset of $\mathcal{M}_+$. For each $x \in \mathcal{M}_h$, we set $\alpha_x := \sup\{\alpha > 0 : x \geq -1/\alpha\}$. Let $\{f_\alpha : \alpha > 0\}$ be as in the proof of Lemma 3.10 and let

$$G := \{x \in \mathcal{M}_h : f_\alpha(x) \in F - \mathcal{M}_+,\ \alpha \in (0, \alpha_x)\}.$$

We first show $G \cap rS$, $r > 0$, is σ-strongly closed. Suppose $\{x_i : i \in I\}$ is a net in $G \cap rS$ converging σ-strongly to some $x \in \mathcal{M}_h$. Since $\|x_i\| < r$ for each $i \in I$, we have $-r \leq x_i$ and hence $\frac{1}{r} \leq \alpha_{x_i}$. Consequently if $0 < \alpha < \frac{1}{2r}$ then $\alpha \in (0, \alpha_{x_i})$ for each $i \in I$ so that by definition of G we have $f_\alpha(x_i) \in F - \mathcal{M}_+$. Thus there exists $\{y_i\}_{i\in I} \subset F$ such that $f_\alpha(x_i) \leq y_i$. But then

$$f_{2\alpha}(x_i) = f_\alpha(f_\alpha(x_i)) \leq f_\alpha(y_i) \leq \frac{1}{\alpha}.$$

As a function on $\mathbb{R}$, for $0 < \alpha < \frac{1}{2r}$ we know $f_{2\alpha}$ is continuous on $[-r, r]$. Hence $\{f_{2\alpha}(x_i) : i \in I\}$ converges to $f_{2\alpha}(x)$ σ-strongly. Since the net $\{f_\alpha(y_i) : i \in I\}$ is uniformly bounded by $\frac{1}{\alpha}$, there exists a subset $\{y_j : j \in J\}$ of $\{y_i\}$ such that $\{f_\alpha(y_j) : j \in J\}$ converges to a $y_\alpha \in \mathcal{M}_h$ σ-weakly. But since $0 \leq f_\alpha(y_j) \leq y_j \in F$ and F is hereditary we know $f_\alpha(y_j) \in F$. But then as F is convex and σ-strongly closed it is also σ-weakly closed and hence we have $y_\alpha \in F$. The inequality

$$y_\alpha - f_{2\alpha}(x) = \lim_{j\in J}(f_\alpha(y_j) - f_{2\alpha}(x_j)) \geq 0,$$

implies that $f_{2\alpha}(x) \in F - \mathcal{M}_+$ for $0 < \alpha < \frac{1}{2r}$; that is, $f_\alpha(x) \in F - \mathcal{M}_+$ if $0 < \alpha < \frac{1}{r}$. To show $x \in G$, it remains to show that we also have $f_\beta(x) \in F - \mathcal{M}_+$ for $\frac{1}{r} \leq \beta < \alpha_x$. Since $f_\alpha(t) \geq f_\beta(t)$ for $0 < \alpha < \beta$ and $t \in \left(-\frac{1}{\beta}, \infty\right)$, we have $f_\beta(x) \leq f_\alpha(x)$. Thus fixing some $\alpha_0 \in (0, 1/r)$ and letting $\beta \in [1/r, \alpha_x)$ we have $f_\beta(x) \leq f_\alpha(x)$ so that

$$f_\beta(x) = f_\alpha(x) - (f_\alpha(x) - f_\beta(x)) \in (F - \mathcal{M}_+) - \mathcal{M}_+ = F - \mathcal{M}_+.$$

Thus $x \in G$ and so $G \cap rS$ is σ-strongly closed.

Next we show
$$G \cap rS = \overline{(F - \mathcal{M}_+) \cap sS)} \cap rS, \ s > r,$$
where the closure is taken in the σ-strong topology. Note that the right hand side of this equation is convex. Suppose $x \in G \cap rS$. Then $f_\alpha(x) \in F - \mathcal{M}_+$ for $0 < \alpha < \alpha_x$ and for sufficiently small α we have $f_\alpha(x) \in sS$. Since $f_\alpha(x) \nearrow x$ as $\alpha \searrow 0$, we have $x \in \overline{(F - \mathcal{M}_+) \cap sS)}$ for any $s > r$. So one containment is clear. Conversely, if $x \in F - \mathcal{M}_+$ then since $f_\alpha(x) \leq x$ for $\alpha \in (0, \alpha_x)$ we have
$$f_\alpha(x) = x - (x - f_\alpha(x)) \in (F - \mathcal{M}_+) - \mathcal{M}_+ = F - \mathcal{M}_+,$$
so that $x \in G$. Hence $F - \mathcal{M}_+ \subset G$ which implies $(E - \mathcal{M}_+) \cap sS \subset G \cap sS$ and consequently $G \cap sS \supset \overline{(F - \mathcal{M}_+) \cap sS)}$ since $G \cap sS$ is σ-strongly closed as shown above. Since $r < s$ we then have
$$G \cap rS = (G \cap sS) \cap rS \supset \overline{(F - \mathcal{M}_+) \cap sS} \cap rS.$$
So we have established the desired equality, which implies $G \cap rS$ is convex. But then G is itself convex and so the σ-strong closedness of $G \cap rS$ implies G is σ-strongly closed. But then $F - \mathcal{M}_+ \subset G \subset \overline{F - \mathcal{M}_+}$ implies
$$G = \overline{F - \mathcal{M}_+}.$$
So if $x \in \overline{F - \mathcal{M}_+} \cap \mathcal{M}_+ = G \cap \mathcal{M}_+$, then $0 \leq f_\alpha(x)$ and $f_\alpha(x) \in F - \mathcal{M}_+$. Since F is hereditary this implies $f_\alpha(x) \in F$ and hence $x = \lim_{\alpha \to 0} f_\alpha(x) \in F$, since F is assumed to be σ-strongly closed. Thus $\overline{F - \mathcal{M}_+} \cap \mathcal{M}_+ \subset F$, and the reverse inclusion is clear. This equality implies condition (i) is the previous lemma is satisfied and hence φ is of the desired form. □

With this result we from now on take normal to mean any of the equivalent conditions in Theorem 3.12. Fix φ and its semi-cyclic representation $\{\pi_\varphi, \mathfrak{H}_\varphi, \eta_\varphi\}$ of $\mathcal{M}$. Set
$$E_\varphi := \bigcup_{\lambda \geq 0} \lambda \Phi_\varphi.$$
For each $\omega \in E_\varphi$, define a sesquilinear form B_ω on $\eta_\varphi(\mathfrak{n}_\varphi)$ by
$$B_\omega(\eta_\varphi(x), \eta_\varphi(y)) = \omega(y^* x), \qquad x, y \in \mathfrak{n}_\varphi.$$

We know $\omega \leq \lambda\varphi$ for some $\lambda > 0$, so B_ω is bounded and hence there exists a unique $h_\omega \in \mathcal{B}(\mathfrak{H}_\varphi)_+$ such that
$$(h_\omega \eta_\varphi(x) \mid \eta_\varphi(y)) = \omega(y^* x), \qquad x, y \in \mathfrak{n}_\varphi.$$
Note that since $\omega(y^* z x) = \omega((z^* y)^* x)$, $h_\omega \in \pi_\varphi(M)'$. Let $\{\pi_\omega, \mathfrak{H}_\omega, \xi_\omega\}$ be the cyclic representation of $\mathcal{M}$ determined by ω via the GNS construction. If we define a map t_ω on $\eta_\varphi(\mathcal{M})$ by
$$t_\omega \eta_\varphi(x) := \pi_\omega(x)\xi_\omega, \qquad x \in \mathfrak{n}_\varphi,$$
then the inequality $\omega \leq \lambda\varphi$ implies t_ω can be extended to all of $\mathfrak{H}_\varphi$ (and is into $\mathfrak{H}_\omega$); we continue to denote this extension by t_ω. Note that
$$(h_\omega \eta_\varphi(x) \mid \eta_\varphi(y)) = \omega(y^* x) = (\pi_\omega(x)\xi_\omega \mid \pi_\omega(y)\xi_\omega) = (t_\omega \eta_\varphi(x) \mid t_\omega \eta_\varphi(y)) = (t_\omega^* t_\omega \eta_\varphi(x) \mid \eta_\varphi(y)),$$
so that $h_\omega = t_\omega^* t_\omega$. Hence the polar decomposition of t_ω looks like
$$t_\omega = u_\omega h_\omega^{\frac{1}{2}}.$$
Also, it is clear that $t_\omega \pi_\varphi(a) = \pi_\omega(a) t_\omega$ for $a \in \mathcal{M}$. So because $h_\omega \in \pi_\varphi(\mathcal{M})'$ we then have
$$u_\omega \pi_\varphi(a) = \pi_\omega(a) u_\omega, \qquad a \in \mathcal{M}.$$
Define
$$\eta_\omega := u_\omega^* \xi_\omega,$$
so that for each $x \in \mathfrak{n}_\varphi$ we have
$$\pi_\varphi(x)\eta_\omega = \pi_\varphi(x) u_\omega^* \xi_\omega = u_\omega^* \pi_\omega(x)\xi_\omega = u_\omega^* t_\omega \eta_\varphi(x) = h_\omega^{\frac{1}{2}} \eta_\varphi(x),$$
or
$$h_\omega^{\frac{1}{2}} d\eta_\varphi(x) = \pi_\varphi(x)\eta_\omega, \qquad x \in \mathfrak{n}_\varphi.$$
Define
$$\mathfrak{p}_{\varphi'} := \{h_\omega \colon \omega \in E_\varphi\} \subset \pi_\varphi(\mathcal{M})'_+.$$

Theorem 3.18. *With the above notation, if*

$$\varphi'(x) := \begin{cases} \|\omega\| & \text{for } x = h_\omega \in \mathfrak{p}_{\varphi'} \\ \infty & \text{otherwise,} \end{cases}$$

then φ' is a faithful semi-finite normal weight on $\pi_\varphi(M)'$.

Proof. It is clear that E_φ is a convex subcone of $\mathcal{M}_*^+$. Also, the map $\omega \mapsto h_\omega$ is homogeneous and additive so we see that $\mathfrak{p}_{\varphi'}$ is convex subcone of $\pi_\varphi(M)'_+$. Towards showing that $\mathfrak{p}_{\varphi'}$ is hereditary, let $x \in \mathcal{M}'$ satisfy $0 \leq x \leq h_\omega$. From Lemma 3.6 we can write $x^{1/2} = s h_\omega^{1/2}$ for $s \in \mathcal{M}'$ with $\|s\| \leq 1$. Define for each $a \in \mathcal{M}$

$$\rho(a) := (\pi_\varphi(a) s\eta_\omega \mid s\eta_\omega).$$

Then for $a, b \in \mathfrak{n}_\varphi$ we have

$$\begin{aligned} \rho(b^*a) &= (\pi_\varphi(b^*a) s\eta_\omega \mid s\eta_\omega) = (\pi_\varphi(a) s\eta_\omega \mid \pi_\varphi(b) s\eta_\omega) = (s\pi_\varphi(a)\eta_\omega \mid s\pi_\varphi(b)\eta_\omega) \\ &= (s h_\omega^{\frac{1}{2}} \eta_\varphi(a) \mid s h_\omega^{\frac{1}{2}} \eta_\varphi(b)) = (x^{\frac{1}{2}} \eta_\varphi(a) \mid x^{\frac{1}{2}} \eta_\varphi(b)) = (x\eta_\varphi(a) \mid \eta_\varphi(b)). \end{aligned}$$

In particular, since $x \leq h_\omega$ we have $\rho(a^*a) = (x\eta_\varphi(a) \mid \eta_\varphi(a)) \leq (h_\omega \eta_\varphi(a) \mid \eta_\varphi(a)) = \omega(a^*a)$. Hence $0 \leq \rho \leq \omega$, which implies $\rho \in E_\varphi$ and $h_\rho = x \in \mathfrak{p}_{\varphi'}$. So $\mathfrak{p}_{\varphi'}$ is hereditary; moreover,

$$\varphi'(x) = \|\rho\| \leq \|\omega\| = \varphi'(h_\omega),$$

so that φ' is well-defined and monotone increasing.

Next we show that φ' is completely additive (i.e. normal). Suppose $h = \sum_{i \in I} h_i$ converges σ-strongly in $\pi_\varphi(\mathcal{M})'_+$. We first consider the case when $\sum_{i \in I} \varphi'(h_i) < \infty$. By the definition of φ' this implies each h_i is of the form $h_i = h_{\omega_i}$ for some $\omega_i \in E_\varphi$. Hence

$$\sum_{i \in I} \|\omega_i\| = \sum_{i \in I} \varphi'(h_i) < \infty,$$

so that $\omega := \sum_{i \in I} \omega_i \in \mathcal{M}_*^+$ converges in norm. Then for each $x \in \mathfrak{n}_\varphi$ we have

$$\begin{aligned} \omega(x^*x) &= \sum_{i \in I} \omega_i(x^*x) = \sum_{i \in I} (h_i \eta_\varphi(x) \mid \eta_\varphi(x)) \\ &= (h\eta_\varphi(x) \mid \eta_\varphi(x)) = \left\| h^{\frac{1}{2}} \eta_\varphi(x) \right\|^2 \leq \|h\| \|\eta_\varphi(x)\|^2 = \|h\| \varphi(x^*x). \end{aligned}$$

Hence $\omega \in E_\varphi$ with $h = h_\omega$, and

$$\varphi'(h) = \|\omega\| = \sum \|\omega_i\| = \sum \varphi'(h_i).$$

Conversely, if $\varphi'(h) < \infty$, then $h = h_\omega$ for some $\omega \in E_\varphi$. Since $\mathfrak{p}_{\varphi'}$ is hereditary (as shown above) we know $h_i \in \mathfrak{p}_{\varphi'}$ for each $i \in I$. For any finite subset $J \subset I$ we have,

$$\sum_{i \in J} \varphi'(h_i) = \varphi' \left(\sum_{i \in J} h_i \right) \leq \varphi'(h) < \infty.$$

So that $\sum_{i \in I} \varphi'(h_i) < \infty$, and through the previous argument we obtain $\varphi'(h) = \sum_{i \in I} \varphi'(h_i)$. Thus φ' is normal.

Since $\|h_\omega\| = 0$ iff $\omega = 0$, φ' is faithful. It remains to show φ' is semi-finite. Let

$$\begin{aligned} \mathfrak{n}_{\varphi'} &:= \{x \in \pi_\varphi(\mathcal{M})' \colon \varphi'(x^*x) < \infty\}; \\ \mathfrak{m}_{\varphi'} &:= \left\{ \sum_{i=1}^n y_i^* x_i \colon x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{n}_{\varphi'} \right\} = \mathfrak{n}_{\varphi'}^* \mathfrak{n}_{\varphi'}. \end{aligned}$$

From Lemma **??**, we know $\mathfrak{p}_{\varphi'} = \mathfrak{m}_{\varphi'}^+$ so that

$$\{h_\omega \colon \omega \in \Phi_\varphi\} = \{x \in \mathfrak{m}_{\varphi'}^+ \colon \|x\| \leq 1\}.$$

But then for each $x \in \mathfrak{n}_\varphi$ we have

$$\begin{aligned} \|\eta_\varphi(x)\|^2 &= \varphi(x^*x) = \sup\{\omega(x^*x) \colon \omega \in \Phi_\varphi\} = \sup\left\{ \|h_\omega^{\frac{1}{2}} \eta_\varphi(x)\|^2 \colon \omega \in \Phi_\varphi \right\} \\ &= \sup\left\{ \|y^{\frac{1}{2}} \eta_\varphi(x)\|^2 \colon y \in \mathfrak{m}_{\varphi'}^+,\ \|y\| \leq 1 \right\}. \end{aligned}$$

Since $\mathfrak{p}_{\varphi'} = \mathfrak{m}_{\varphi'}^+$, $y \in \mathfrak{m}_{\varphi'}^+$ with $\|y\| \leq 1$ can be written as $y = h^2$ for $h \in \mathfrak{n}_{\varphi'}$ with $0 \leq h \leq 1$. Hence we obtain

$$\|\eta_\varphi(x)\|^2 = \sup\left\{\|h\eta_\varphi(x)\|^2 \colon h \in \mathfrak{n}_{\varphi'},\ 0 \leq h \leq 1\right\},$$

which implies that $\mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'}^*$ is non-degenerate on $\mathfrak{H}_\varphi$. [WUT?:] Thus the open unit ball of $(\mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'}^*)_+$ is upward directed and converges to the identity σ-strongly. Since $a^*\pi_\varphi(\mathcal{M})'a \subset \mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'}^*$ for every $a \in \mathfrak{n}_{\varphi'}$ (because for $x \in \pi_\varphi(\mathcal{M})'$ we have $(a^*xa)^*a^*xa = a^*x^*aa^*xa \leq \|x\|^2\|a\|^2a^*a$ and $a^*xa(a^*xa)^* = a^*xaa^*x^*a \leq \|x\|^2\|a\|^2a^*a$), $\pi_\varphi(\mathcal{M})'$ is the σ-strong closure of $\mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'}^*$. Thus φ' is semi-finite on $\pi_\varphi(\mathcal{M})'$. □

Definition 3.19. The weight φ' on $\pi_\varphi(\mathcal{M})'$ is called the **opposite weight** of φ.

Given a normal weight φ on $\mathcal{M}$, let e be the projection in $\mathcal{M}$ such that $\mathcal{M}e = \bar{\mathfrak{n}_\varphi}$, the σ-strong closure, and let f be the projection in $\mathcal{M}$ such that $\mathcal{M}f = \{x \in \mathcal{M} \colon \varphi(x^*x) = 0\}$. Then φ is semi-finite on $e\mathcal{M}e$ and faithful on $(1-f)\mathcal{M}(1-f)$. The projection $s(\varphi) := e - f$ is the called the support of φ.

Hence forth any weight on a von Neumann algebra is assumed to be semi-finite and normal.

4. Left Hilbert Algebras to Weights and Back Again

In this section we explore the correspondence between the full left Hilbert algebras of Section 1 and the weights of the previous section.

Let $\mathfrak{A}$ be a full left Hilbert algebra with completion $\mathfrak{H}$ and left von Neumann algebra $\mathcal{M} = \mathcal{R}_l(\mathfrak{A})$. Recall

$$\pi_l(\mathfrak{A}) = \mathfrak{n}_l \cap \mathfrak{n}_l^*, \qquad \pi_r(\mathfrak{A}') = \mathfrak{n}_r \cap \mathfrak{n}_r^*.$$

We then define

$$\mathfrak{m}_l := \mathfrak{n}_l^*\mathfrak{n}_l, \qquad \text{and} \qquad \mathfrak{m}_r := \mathfrak{n}_r^*\mathfrak{n}_r.$$

Next we will define a positive extended-real valued function φ_l on $\mathcal{M}_+$ as follows:

$$\varphi_l(x) := \begin{cases} \|\xi\|^2 & \text{if } x^{\frac{1}{2}} = \pi_l(\xi),\ \xi \in \mathfrak{A}; \\ \infty & \text{otherwise.} \end{cases}$$

Similarly, we define φ_r on $\mathcal{M}_+'$:

$$\varphi_r(y) := \begin{cases} \|\eta\|^2 & \text{if } y^{\frac{1}{2}} = \pi_r(\eta),\ \eta \in \mathfrak{A}'; \\ \infty & \text{otherwise.} \end{cases}$$

We work towards showing that these are semi-finite normal weights.

Lemma 4.1. *In the above situation, $\mathfrak{m}_l^+$ (resp. $\mathfrak{m}_r^+$) is a hereditary convex subcone of $\mathcal{M}_+$ (resp. $\mathcal{M}_+'$). Furthermore, $\mathfrak{n}_l$ and $\mathfrak{n}_r$ are given by the following:*

$$\mathfrak{n}_l = \{x \in \mathcal{M} \colon x^*x \in \mathfrak{m}_l^+\}, \qquad \mathfrak{n}_r = \{y \in \mathcal{M}' \colon y^*y \in \mathfrak{m}_r^+\}.$$

Proof. $\mathfrak{m}_l$ is a self-adjoint subalgebra of $\mathcal{M}$, so $\mathfrak{m}_l^+$ is a convex subcone of $\mathcal{M}_+$. Suppose we have $0 \leq b \leq a$ for $a \in \mathfrak{m}_l^+$ and $b \in \mathcal{M}_+$. Writing $a = \sum_{i=1}^n x_i^*y_i$ for $x_i, y_i \in \mathfrak{n}_l$ we have

$$a = \frac{1}{2}(a + a^*) = \sum_{i=1}^n \frac{1}{2}(x_i^*y_i + y_i^*x_i) \leq \sum_{i=1}^n (x_i^*x_i + y_i^*y_i) \in \mathfrak{m}_l^+.$$

So upon replacing a with the last expression above we may assume b is dominated by $\sum_{i=1}^n x_i^*x_i$. By definition of $\mathfrak{n}_l$, $x_i = \pi_l(\xi_i)$ with $\xi_i \in \mathfrak{B}$ for each i. Using Lemma 3.6.(ii), we can find $s_1, \ldots, s_n$ in the unit ball of $\mathcal{M}$ so that $x_i^{1/2} = s_ia^{1/2}$ and $p = \sum_{i=1}^n s_i^*s_i$ is the range projection $s(a)$ of a. Set $\xi = \sum_{i=1}^n s_i^*\xi_i$. Then $\xi \in \mathfrak{B}$ and

$$\pi_l(\xi) = \sum_{i=1}^n s_i^*\pi_l(\xi_i) = \sum_{i=1}^n s_i^*x_i = \sum_{i=1}^n s_i^*s_ia^{\frac{1}{2}} = pa^{\frac{1}{2}} = a^{\frac{1}{2}}.$$

Hence $a^{1/2} = \pi_l(\xi)$. Using Lemma 3.6.(i), choose $s \in \mathcal{M}$ so that $b^{1/2} = sa^{1/2}$. Then

$$b^{\frac{1}{2}} = s\pi_l(\xi) = \pi_l(s\xi) \in \mathfrak{n}_l.$$

Hence $b = b^{1/2}b^{1/2} \in \mathfrak{m}_l^+$, and thus $\mathfrak{m}_l^+$ is hereditary.

Now, let $x \in \mathfrak{n}_r$, then $x^*x \in \mathfrak{m}_l^+$ by definition of $\mathfrak{m}_l$. Conversely, if $x^*x \in \mathfrak{m}_l^+$, then $|x| = (x^*x)^{1/2}$ is of the form $|x| = \pi_l(\xi)$ for some $\xi \in \mathfrak{B}$ as we showed above (i.e. let $a = |x|$). So if $x = u|x|$ is the polar decomposition of x, then we have $x = u\pi_l(\xi) = \pi_l(u\xi) \in \mathfrak{n}_l$.

By symmetry we have the same result for $\mathfrak{n}_r$. □

For each $\xi \in \mathfrak{H}$ we define $\omega^l_\xi \in \mathcal{M}^+_*$ and $\omega^r_\xi \in (\mathcal{M}')^+_*$ by

$$\omega^l_\xi(x) = (x\xi \mid \xi),\ x \in \mathcal{M}; \qquad \omega^r_\xi(y) = (y\xi \mid \xi),\ y \in \mathcal{M}'.$$

Consider the following sets:

$$\begin{aligned}\Phi_l &:= \left\{\omega^l_\eta \colon \eta \in \mathfrak{B}',\ \|\pi_r(\eta)\| \le 1\right\}\\ \Phi_{l,0} &:= \left\{\omega^l_\eta \colon \eta \in \mathfrak{B}',\ \|\pi_r(\eta)\| < 1\right\}\\ \Phi_r &:= \left\{\omega^r_\xi \colon \xi \in \mathfrak{B},\ \|\pi_l(\xi)\| \le 1\right\}\\ \Phi_{r,0} &:= \left\{\omega^r_\xi \colon \xi \in \mathfrak{B},\ \|\pi_l(\xi)\| < 1\right\}\end{aligned}$$

Lemma 4.2. *Let S and S_0 (resp. S' and S'_0) denote the closed and open unit balls of $\mathcal{M}$ (resp. $\mathcal{M}'$). There exists a completely positive map θ (resp. θ') from $\mathfrak{m}_l$ into $\mathcal{M}'_*$ (resp. $\mathfrak{m}_r$ into $\mathcal{M}_*$) such that*

$$\begin{aligned}\theta(\pi_l(\xi)^*\pi_l(\xi)) &= \omega^r_\xi, \qquad \xi \in \mathfrak{B}\\ \theta'(\pi_r(\eta)^*\pi_r(\eta)) &= \omega^l_\eta, \qquad \eta \in \mathfrak{B}'\end{aligned}$$

and such that

$$\begin{aligned}\theta(\mathfrak{m}^+_l \cap S) &= \Phi_r, \qquad \theta(\mathfrak{m}^+_r \cap S_0) = \Phi_{r,0}.\\ \theta'(\mathfrak{m}^+_r \cap S') &= \Phi_l, \qquad \theta'(\mathfrak{m}^+_l \cap S'_0) = \Phi_{l,0}.\end{aligned}$$

Proof. The existence of θ and θ' along with their formulas follow from a proof analogous to the one in Lemma 3.8. Then the set equalities are obvious from the definitions. □

Corollary 4.3. *The sets $\Phi_{l,0}$ and $\Phi_{r,0}$ are hereditary convex subsets of $\mathcal{M}^+_*$ and $(\mathcal{M}')^+_*$ respectively.*

Lemma 4.4. *The functions φ_l and φ_r are given by*

$$\begin{aligned}\varphi_l(x) &= \sup\{\omega(x) \colon \omega \in \Phi_{l,0}\}, \qquad x \in \mathcal{M}_+\\ \varphi_r(x) &= \sup\{\omega(y) \colon \omega \in \Phi_{r,0}\}, \qquad y \in \mathcal{M}'_+.\end{aligned}$$

Proof. We establish the formula for φ_l and the other to symmetry. Let ψ be the function defined by the right hand side of the first equation.

We know from Lemma 4.2 that $\Phi_{l,0}$ is upward directed, and hence ψ is a normal weight on $\mathcal{M}_+$. Given $a \in \mathfrak{m}^+_l$ we can write $a^{1/2} = \pi_l(\xi)$ with $\xi \in \mathfrak{A}$. We compute:

$$\begin{aligned}\psi(a) &= \sup\{\omega(a) \colon \omega \in \Phi_{l,0}\} = \sup\{(a\eta \mid \eta) \colon \eta \in \mathfrak{B}',\ \|\pi_r(\eta)\| < 1\}\\ &= \sup\left\{\left\|a^{\frac{1}{2}}\eta\right\|^2 \colon \eta \in \mathfrak{B}',\ \|\pi_r(\eta)\| < 1\right\} = \sup\{\|\pi_l(\xi)\eta\|^2 \colon \eta \in \mathfrak{B}',\ \|\pi_r(\eta)\| < 1\}\\ &= \sup\{\|\pi_r(\eta)\xi\|^2 \colon \eta \in \mathfrak{B}',\ \|\pi_r(\eta)\| < 1\} = \sup\{\|b\xi\|^2 \colon b \in S'_0 \cap \mathfrak{n}_r\} = \|\xi\|^2 = \varphi_l(a),\end{aligned}$$

where we have used in the second to last equality the fact that $S'_0 \cap \mathfrak{n}^+_r$ is upward directly and converges strongly to the identity. Conversely, suppose $\psi(a) = \lambda < \infty$, $a \in \mathcal{M}_+$. We need to show $a \in \mathfrak{m}^+_l$ in which case the above argument will apply $\psi(a) = \varphi_l(a)$. Define ω'_a on $\mathfrak{m}_r$ as follows:

$$\omega'_a(y) := \langle a, \theta'(y)\rangle, \qquad y \in \mathfrak{m}_r.$$

Since θ' is completely positive, ω'_a is positive and

$$\|\omega'_a\| = \sup\{\omega'_a(y) \colon y \in \mathfrak{m}^+_r \cap S'_0\} = \sup\{\langle a, \theta'(y)\rangle : y \in \mathfrak{m}^+_r \cap S'_0\} = \sup\{\langle a, \omega\rangle : \omega \in \Phi_{l,0}\} = \psi(a) = \lambda < \infty.$$

Hence ω'_a is bounded and we can extend it to the norm closure A_r of $\mathfrak{m}_r$ as a positive linear functional, which we continue denote ω'_a. For $y \in A_r$ we have

$$|\omega'_a(y)|^2 \le \|\omega'_a\|\omega'_a(y^*y) = \lambda\omega'_a(y^*y).$$

Hence for $\eta \in \mathfrak{B}'$ we have

$$|\omega'_a(\pi_r(\eta))| \le \sqrt{\lambda}\omega'_a(\pi_r(\eta)^*\pi_r(\eta))^{\frac{1}{2}} = \sqrt{\lambda}(a\eta \mid \eta)^{\frac{1}{2}} = \sqrt{\lambda}\left\|a^{\frac{1}{2}}\eta\right\|.$$

By the Riesz representation theorem, there exists a vector $\xi \in \left[a^{1/2}\mathfrak{H}\right]$ such that

$$\omega_a'(\pi_r(\eta)) = (a^{\frac{1}{2}}\eta \mid \xi), \qquad \eta \in \mathfrak{B}'.$$

Then for each $\eta, \zeta \in \mathfrak{B}'$ we have

$$(a^{\frac{1}{2}}\eta \mid a^{\frac{1}{2}}\zeta) = (a\eta \mid \zeta) = \omega_a'(\pi_r(\zeta)^*\pi_r(\eta)) = (a^{\frac{1}{2}}\pi_r(\zeta)^*\eta \mid \xi) = (a^{\frac{1}{2}}\eta \mid \pi_r(\zeta)\xi).$$

But since

$$\pi_r(\zeta)\xi \in \pi_r(\zeta)[a^{\frac{1}{2}}\mathfrak{H}] \subset [a^{\frac{1}{2}}\pi_r(\zeta)\mathfrak{H}] \subset [a^{\frac{1}{2}}\mathfrak{H}],$$

this impiles $a^{1/2}\zeta = \pi_r(\zeta)\xi$ for $\zeta \in \mathfrak{B}'$. Hence ξ is left bounded and $a^{1/2} = \pi_l(\xi) \in \mathfrak{n}_l$. Consequently $a = (a^{1/2})^2 \in \mathfrak{m}_l^+$ and $\varphi_l(a) = \psi(a)$ by the previous argument. □

We establish the first of our two main goals in the section.

Theorem 4.5. *If $\mathfrak{A}$ is a full left Hilbert algebra with left von Neumann algebra $\mathcal{M} = \mathcal{R}_l(\mathfrak{A})$, then φ_l and φ_r defined above give faithful semi-finite normal weights on $\mathcal{M}$ and $\mathcal{M}'$, respectively, with the following properties:*

(i) *The action of $\mathcal{M}$ on $\mathfrak{H}$, the completion of $\mathfrak{A}$, is unitary equivalent to the semi-cyclic representation π_{φ_l} of $\mathcal{M}$ on $\mathfrak{H}_{\varphi_l}$ under the correspondence:*

$$U\xi = \eta_{\varphi_l}(\pi_l(\xi)), \qquad \xi \in \mathfrak{B}.$$

(ii) *Identifying $\mathfrak{H}$ and $\mathfrak{H}_{\varphi_l}$ under the unitary U above, φ_r is the opposite weight of φ_l.*

(iii)

$$\begin{aligned} \mathfrak{m}_{\varphi_l} = \mathfrak{m}_l, &\qquad \mathfrak{n}_{\varphi_l} = \mathfrak{n}_l, \\ \mathfrak{m}_{\varphi_r} = \mathfrak{m}_r, &\qquad \mathfrak{n}_{\varphi_r} = \mathfrak{n}_r, \\ \varphi_l(\pi_l(\eta)^*\pi_l(\xi)) =&(\xi \mid \eta), \qquad \xi, \eta \in \mathfrak{B} \\ \varphi_r(\pi_r(\eta)^*\pi_r(\xi)) =&(\xi \mid \eta), \qquad \xi, \eta \in \mathfrak{B}'. \end{aligned}$$

Proof. The previous lemma shows that φ_l and φ_r are normal weights. Also, since $\mathfrak{m}_l^+$ generates $\mathcal{M}$, φ_l is semi-finite. To reduce the number of subscripts, denote φ_l by φ. Let $\{\pi_\varphi, \mathfrak{h}_\varphi, \eta_\varphi\}$ be the semi-cyclic representation induced by φ. If $\xi_1, \xi_2 \in \mathfrak{B}$, then

$$(\xi_1 \mid \xi_2) = \varphi(\pi_l(\xi_2)^*\pi_l(\xi_1)) = (\eta_\varphi(\pi_l(\xi_1)) \mid \eta_\varphi(\pi_l(\xi_2)))$$

follows from the polarization identity applied to the definition of $\varphi = \varphi_l$. Hence the map U in the statement of the theorem can be extended to an isometry from $\mathfrak{H}$ onto $\mathfrak{H}_\varphi$, still denoted U. Also, it is clear that $UaU^* = \pi_\varphi(a)$ for $a \in \mathcal{M}$. If we identify $\mathfrak{H}_\varphi$ with $\mathfrak{H}$ by U, then the above equation shows that π_l and η_φ are inverses of each other. □

We now proof the converse to the previous theorem:

Theorem 4.6. *Let φ be a faithful semi-finite normal weight on a von Neumann algebra $\mathcal{M}$. Let*

$$\mathfrak{A}_\varphi = \eta_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*).$$

If we define an involutive algebra structure in $\mathfrak{A}_\varphi$ in the following fashion:

$$\begin{aligned} \eta_\varphi(x)\eta_\varphi(y) = \eta_\varphi(xy) &\qquad x, y \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*, \\ \eta_\varphi(x)^\sharp = \eta_\varphi(x^*), \end{aligned}$$

then $\mathfrak{A}_\varphi$ is a full left Hilbert algebra such that

(i) *$\pi_\varphi(\mathcal{M}) = \mathcal{R}_l(\mathfrak{A}_\varphi)$;*

(ii) *if we identify $\mathcal{M}$ and $\mathcal{R}_l(\mathfrak{A}_\varphi)$ via π_φ, then the weight φ_l associated with $\mathfrak{A}_\varphi$ agrees with the original weight φ.*

Proof. If $x, y \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$, then $\eta_\varphi(x)\eta_\varphi(y) = \eta_\varphi(xy) = \pi_\varphi(x)\eta_\varphi(y)$, so that multiplication is left continuous and $\pi_l(\eta_\varphi(x)) = \pi_\varphi(x)$. Hence $\pi_l(\mathfrak{A}_\varphi) = \pi_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ so that $\pi_l(\mathfrak{A}_\varphi)$ generates $\pi_\varphi(\mathcal{M})$ as a von Neumann algebra. Consequently, $\pi_l(\mathfrak{A}_\varphi)$ is non-degenerate on $\mathfrak{H}_\varphi$ and therefore $\mathfrak{A}_\varphi^2$ is dense in $\mathfrak{H}_\varphi$ since $\pi_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ contains an increasing net converging strongly to 1. Note that for $x, y, z \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ we have

$$(\eta_\varphi(x)\eta_\varphi(y) \mid \eta_\varphi(z))_\varphi = \varphi(z^*xy) = \varphi((x^*z)^*y) = (\eta_\varphi(y) \mid \eta_\varphi(x^*)\eta_\varphi(z))_\varphi = (\eta_\varphi(y) \mid \eta_\varphi(x)^\sharp\eta_\varphi(z))_\varphi.$$

It remains to show condition (c) in Definition ??.

Set

$$\Phi_{\varphi,0} := \left\{\omega \in \mathcal{M}_*^+ \colon (1+\epsilon)\omega \le \varphi \text{ for some } \epsilon > 0\right\}.$$

Recall that from the discussion following the proof of Theorem 3.12 that for each $\omega \in \Phi_{\varphi,0}$ there corresponds an $h_\omega \in \pi_\varphi(\mathcal{M})'_+$ with $\|h_\omega\| < 1$ and an $\eta_\omega \in \mathfrak{H}_\varphi$ such that

$$\omega(x) = (\pi_\varphi(x)\eta_\omega \mid \eta_\omega) \qquad x \in \mathcal{M}, \qquad h_\omega^{\frac{1}{2}}\eta_\varphi(x) = \pi_\varphi(x)\eta_\omega \qquad x \in \mathfrak{n}_\varphi.$$

Now, for each $\omega_1, \omega_2 \in \Phi_{\varphi,0}$, $b \in \pi_\varphi(\mathcal{M})'$ and $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$, we compute:

$$\begin{aligned}(\eta_\varphi(x^*) \mid h_{\omega_1}^{\frac{1}{2}} b\eta_{\omega_2}) &= (h_{\omega_1}^{\frac{1}{2}}\eta_\varphi(x^*) \mid b\eta_{\omega_2}) = (\pi_\varphi(x)^*\eta_{\omega_1} \mid b\eta_{\omega_2}) = (\eta_{\omega_1} \mid \pi_\varphi(x) b\eta_{\omega_2}) \\ &= (\eta_{\omega_1} \mid b\pi_\varphi(x)\eta_{\omega_2}) = (\eta_{\omega_1} \mid b h_{\omega_2}^{\frac{1}{2}}\eta_\varphi(x)) = (h_{\omega_2}^{\frac{1}{2}} b^*\eta_{\omega_1} \mid \eta_\varphi(x)).\end{aligned}$$

Suppose $\{x_n\}$ is a sequence in $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ such that

$$\lim_{n\to\infty} \eta_\varphi(x_n) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \eta_\varphi(x_n^*) = \xi \in \mathfrak{H}_\varphi$$

in norm, then the above computation shows that

$$(\xi \mid h_{\omega_1}^{\frac{1}{2}} b\eta_{\omega_2}) = \lim_{n\to\infty} (\eta_\varphi(x_n^*) \mid h_{\omega_1}^{\frac{1}{2}} b\eta_{\omega_2}) = lim_{n\to\infty} (h_{\omega_2}^{\frac{1}{2}} b^*\eta_{\omega_1} \mid \eta_\varphi(x_n)) = 0$$

for every $b \in \pi_\varphi(\mathcal{M})'$ and $\omega_1, \omega_2 \in \Phi_{\varphi,0}$. Hence to show $\xi = 0$ it suffices to prove the linear span of

$$\bigcup \left\{h_{\omega_1}^{\frac{1}{2}}\pi_\varphi(\mathcal{M})'\eta_{\omega_2} \colon \omega_1, \omega_2 \in \Phi_{\varphi,0}\right\}$$

is dense in in $\mathfrak{H}_\varphi$. We compute for $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$:

$$\begin{aligned}\|\eta_\varphi(x)\|^2 = \varphi(x^*x) &= \sup\{\omega(x^*x) \colon \omega \in \Phi_{\varphi,0}\} = \sup\left\{\|\pi_\varphi(x)\eta_\omega\|^2 \colon \omega \in \Phi_{\varphi,0}\right\} \\ &= \sup\left\{\left\|h_\omega^{\frac{1}{2}}\eta_\varphi(x)\right\|^2 \colon \omega \in \Phi_{\varphi,0}\right\} = \sup\{(h_\omega\eta_\varphi(x) \mid \eta_\varphi(x)) \colon \omega \in \Phi_{\varphi,0}\}.\end{aligned}$$

Since $\Phi_{\varphi,0}$ is convex and the map: $\omega \mapsto h_\omega \in \pi_\varphi(\mathcal{M})'_+$ is affine, 1 is in the strong closure of $\{h_\omega \colon \omega \in \Phi_{\varphi,0}\}$ [WHY?]. Hence $\bigcup\left\{h_{\omega_1}^{\frac{1}{2}}\pi_\varphi(\mathcal{M})'\eta_{\omega_2} \colon \omega_1 \in \Phi_{\varphi,0}\right\}$ is dense in $\pi_\varphi(\mathcal{M})'\eta_{\omega_2}$.

Now we prove that $\bigcup\{\pi_\varphi(\mathcal{M})'\eta_\omega \colon \omega \in \Phi_{\varphi,0}\}$ is total in $\mathfrak{H}_\varphi$. Set $\mathfrak{R}$ to be the closure of the span of this union. Then let e be the projection of $\mathfrak{H}_\varphi$ onto $\mathfrak{R}$. By definition, $\mathfrak{R}$ is invariant under $\pi_\varphi(\mathcal{M})'$, hence $e \in \pi_\varphi(\mathcal{M})$ and $(1-e)\eta_\omega = 0$ for every $\omega \in \Phi_{\varphi,0}$. Let $f \in \mathrm{Proj}(\mathcal{M})$ be such that $\pi_\varphi(f) = 1 - e$. Then

$$\varphi(f) = \sup\{\omega(f) \colon \omega \in \Phi_{\varphi,0}\} = \sup\{(\pi_\varphi(f)\eta_\omega \mid \eta_\omega) \colon \omega \in \Phi_{\varphi,0}\} = \sup\{((1-e)\eta_\omega \mid \eta_\omega) \colon \omega \in \Phi_{\varphi,0}\} = 0.$$

Recalling that φ is faithful we see that $f = 0$ and hence $e = 1$, ergo $\mathfrak{R} = \mathfrak{H}_\varphi$ and so the desired density is established. Hence the $\sharp$-operation is preclosed and $\mathfrak{A}_\varphi$ is a left Hilbert algebra.

We next show that $\mathfrak{A}_\varphi$ is full. From the equality $\pi_\varphi(x)\eta_\omega = h_\omega^{\frac{1}{2}}\eta_\varphi(x)$ we see that η_ω is right bounded for each $\omega \in \Phi_{\varphi,0}$ and $\pi_r(\eta_\omega) = h_\omega^{\frac{1}{2}}$. Since $h_\omega^{\frac{1}{2}}$ is self-adjoint, $\eta_\omega \in \mathfrak{A}'_\varphi$ by Lemma 1.17. Set $x = \pi_l(\xi) \in \pi_\varphi(\mathcal{M})$ for a left bounded vector $\xi \in \mathfrak{H}_\varphi$ (so that $\eta_\varphi(x) = \xi$) and compute:

$$\begin{aligned}\varphi(x^*x) &= \sup\{\omega(x^*x) \colon \omega \in \Phi_{\varphi,0}\} = \sup\left\{\|x\eta_\omega\|^2 \colon \omega \in \Phi_{\varphi,0}\right\} \\ &= \sup\left\{\left\|h_\omega^{\frac{1}{2}}\eta_\varphi(x)\right\|^2 \colon \omega \in \Phi_{\varphi,0}\right\} = \|\eta_\varphi(x)\|^2 = \|\xi\|^2 < \infty.\end{aligned}$$

Hence $x \in \mathfrak{n}_\varphi$ and $\varphi(x^*x) = \|\xi\|^2$. Thus $\pi_l(\mathfrak{B}_\varphi) \subset \pi_\varphi(\mathfrak{n}_\varphi)$, where $\mathfrak{B}_\varphi$ is the set of all left bounded vectors in $\mathfrak{H}_\varphi$. So from Lemma 1.17$'$.(ii$'$) we have

$$\pi_l(\mathfrak{A}_\varphi) \subset \pi_l(\mathfrak{A}''_\varphi) = \pi_l(\mathfrak{B}_\varphi) \cap \pi_l(\mathfrak{B}_\varphi)^* \subset \pi_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*) = \pi_l(\mathfrak{A}_\varphi),$$

so that $\mathfrak{A}_\varphi = \mathfrak{A}''_\varphi$. Hence $\mathfrak{A}_\varphi$ is full.

Finally, if $x = uh \in \mathfrak{n}_\varphi$ is the polar decomposition, then $h = u^*x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$, so that $\eta_\varphi(x) = \pi_\varphi(u)\eta_\varphi(h) \in \pi_\varphi(u)\mathfrak{A}_\varphi \subset \mathfrak{B}_\varphi$. Thus $\eta_\varphi(x)$ is left bounded and $\pi_l(\eta_\varphi(x)) = \pi_\varphi(x)$. Thus $\pi_l(\mathfrak{B}_\varphi) = \pi_\varphi(\mathfrak{n}_\varphi)$. Now, for every $x \in \mathfrak{n}_\varphi = \pi_\varphi^{-1}(\mathfrak{n}_l)$,

$$\varphi(x^*x) = \|\eta_\varphi(x)\|^2 = \varphi_l\left(\pi_l(\eta_\varphi(x))^*\pi_l(\eta_\varphi(x))\right) = \varphi_l(\pi_\varphi(x)^*\pi_\varphi(x))$$

(the second equality follows from the definition of φ_l). Thus $\varphi = \varphi_l \circ \pi_\varphi$ so that φ is identified with φ_l via π_φ. $\square$

Lastly we convince the reader of the relevance of the above to any von Neumann algebra:

Theorem 4.7. *Every von Neumann algebra admits a faithful semi-finite normal weight.*

Proof. Let $\{\omega_i \colon i \in I\}$ be a maximal family of normal positive linear functionals on a given von Neumann algebra $\mathcal{M}$ with orthogonal support $\{s(\omega_i)\}$. By the maximality, we have $\sum_{i\in I} s(\omega_i) = 1$. We then set

$$\varphi(x) = \sum_{i\in I} \omega_i(x), \qquad x \in \mathcal{M}_+.$$

Let $J \subset\subset I$ mean that J is a finite subset of I, then

$$\varphi(x) = \sup\left\{\sum_{i\in J} \omega_i(x) \colon J \subset\subset I\right\}.$$

Hence φ is a normal weight. For each $J \subset\subset I$, set $p_J = \sum_{i\in J} s(\omega_i) \in \mathrm{Proj}(\mathcal{M})$. Then $p_J \in \mathfrak{m}_\varphi^+$ and $p_J \nearrow 1$, so that φ is semi-finite. If $\varphi(x) = 0$ for $x \in \mathcal{M}_+$, then $\omega_i(x) = 0$ for every $i \in I$, so that $x^{1/2}s(\omega_i) = 0$, $i \in I$; hence $x^{1/2} = x^{1/2}\sum s(\omega_i) = \sum x^{1/2}s(\omega_i) = 0$. Thus φ is faithful. $\square$

5. Modular Automorphism Group of a Weight

After way too many pages of what were essentially "preliminaries" we arrive what could be (but shouldn't be) described as the "meat and potatoes" of the theory.

Throughout this section all weights are assumed to be faithful, semi-finite, and normal unless stated otherwise.

Given the previous section we now know how to build a left Hilbert algebra from a pair $(\mathcal{M}, \varphi)$ consisting of a von Neumann algebra and a weight. Then, given Section 1 we can consider the modular operator Δ associated to this left Hilbert algebra. Presently we explore the connection between φ and Δ.

Definition 5.1. Let A be a C^*-algebra, equipped with a one parameter automorphism group $\{\sigma_t \colon t \in \mathbb{R}\}$. A lower semi continuous weight φ on A is said to satisfy the **modular condition** for $\{\sigma_t\}$ if the following two conditions hold:

(i) $\varphi = \varphi \circ \sigma_t,\ t \in \mathbb{R}$;

(ii) For every pair $x, y \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$, there exists a bounded continuous function $F_{x,y}$ on the closed horizontal strip $\bar{\mathbb{D}}$ and holomorphic on the open strip $\mathbb{D}$ (where $\mathbb{D}$ is bounded by $\mathbb{R}$ and $\mathbb{R} + i$) such that

$$F_{x,y}(t) = \varphi(\sigma_t(x)y) \qquad \text{and} \qquad F_{x,y}(t+i) = \varphi(y\sigma_t(x)), \qquad t \in \mathbb{R}.$$

Theorem 5.2. *To each weight φ on a von Neumann algebra $\mathcal{M}$ there corresponds uniquely a one parameter automorphism group $\{\sigma_t\}$ of $\mathcal{M}$ for which φ satisfies the modular condition.*

Proof. Let $\mathfrak{A}$ be the full left Hilbert algebra corresponding to φ, guaranteed by Theorem 4.6. Using the semi-cyclic representation $\{\pi_\varphi, \mathfrak{H}_\varphi, \eta_\varphi\}$ we identify $\mathcal{M}$ with $\pi_\varphi(\mathcal{M}) = \mathcal{R}_l(\mathfrak{A})$. From Theorem 1.24 the modular operator Δ of $\mathfrak{A}$ gives rise to a one parameter automorphism group $\{\sigma_t\}$ of $\mathcal{M}$ by the following:

$$\sigma_t(x) = \Delta^{it} x \Delta^{-it}, \qquad x \in \mathcal{M}.$$

Since $\mathfrak{A} = \eta_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ and the $\{\Delta^{it}\}$ act on $\mathfrak{A}$ as automorphisms (by Theorem 1.24) it easy to see that $\sigma_t(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*) = \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ and hence σ_t leaves $\mathfrak{m}_\varphi$ globally invariant. Now, for $\xi, \eta \in \mathfrak{A}$ we have

$$\varphi(\sigma_t(\pi_l(\eta)^*\pi_l(\xi))) = \varphi(\pi_l(\Delta^{it}\eta)^*\pi_l(\Delta^{it}\xi)) = (\Delta^{it}\xi \mid \Delta^{it}\eta) = (\xi \mid \eta) = \varphi(\pi_l(\eta)^*\pi_l(\xi)).$$

Hence $\varphi\,|_{\pi_l(\mathfrak{A}^2)}$ is invariant under $\{\sigma_t\}$ and by density φ itself is as well.

For $\xi, \eta \in \mathfrak{A}$ set $x = \pi_l(\xi)$ and $y = \pi_l(\eta)$. We define the following function:

$$F(\alpha) = \left(\delta^{-\frac{i\alpha}{2}}\xi \mid \Delta^{\frac{i\bar{\alpha}}{2}}\eta\right).$$

Since $\xi, \eta \in \mathfrak{A} \subset \mathfrak{D}^\sharp$, the vector valued functions:

$$\xi(\alpha) = \Delta^{-\frac{i\alpha}{2}}\xi, \qquad \eta(\alpha) = \Delta^{-\frac{i\alpha}{2}}\eta$$

are both, by Lemma 2.3, bounded holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Hence F is holomorphic in $\mathbb{D}$ and bounded continuous on $\overline{\mathbb{D}}$. We compute:

$$\begin{aligned} F(t) &= (\Delta^{-\frac{it}{2}}\xi \mid \Delta^{\frac{it}{2}}\eta) = (\xi \mid \Delta^{it}\eta) = \varphi(\pi_l(\Delta^{it}\eta)^*\pi_l(\xi)) = \varphi(\sigma_t(y^*)x); \\ F(t+i) &= (\Delta^{-\frac{it}{2}}\Delta^{\frac{1}{2}}\xi \mid \Delta^{\frac{it}{2}}\Delta^{\frac{1}{2}}\eta) = (\Delta^{\frac{1}{2}}\xi \mid \Delta^{\frac{1}{2}}\Delta^{it}\eta) = (J\Delta^{\frac{1}{2}}\Delta^{it}\eta \mid J\Delta^{\frac{1}{2}}\xi) \\ &= ((\Delta^{it}\eta)^\sharp \mid \xi^\sharp) = \varphi(\pi_l(\xi)\pi_l(\Delta^{it}\eta)^*) = \varphi(x\sigma_t(y^*)). \end{aligned}$$

Hence $\{\sigma_t\}$ satisfies the modular condition.

It remains to show the uniqueness of $\{\sigma_t\}$ so suppose φ satisfies the modular condition for another one parameter automorphism group $\{\sigma'_t\}$ of $\mathcal{M}$. Since φ is invariant under $\{\sigma'_t\}$ there exists a one parameter unitary group $\{U(t)\}$ on $\mathfrak{A}$ satisfying

$$U(t)\eta_\varphi(x) = \eta_\varphi(\sigma'_t(x)), \qquad x \in \mathfrak{n}_\varphi,\ t \in \mathbb{R}.$$

Recall that [FIND A PLACE TO PUT FORMAL DEFINITION OF ONE PARAMETER UNITARY GROUP] there is an implicit continuity condition when we speak of one parameter unitary groups, but in our situation this follows from the continuity of the functions $F_{x,y}$ in Definition 5.1. By Stone's Theorem, there exists a self-adjoint operator K such that $U(t) = \exp \imath tK$. Set $H := \exp K$, then we want to prove that $H = \Delta$, so that

$$\eta_\varphi(\sigma'_t(x)) = U(t)\eta_\varphi(x) = H^{\imath t}\eta_\varphi(x) = \Delta^{\imath t}\eta_\varphi(x) = \eta_\varphi(\sigma_t(x)),$$

and hence $\sigma_t(x) = \sigma'_t(x)$ for $x \in \mathfrak{n}_\varphi$. Since $\mathfrak{n}_\varphi$ generates $\mathcal{M}$ we'll then know that $\sigma_t = \sigma'_t$ for all $t \in \mathbb{R}$.

Now, $\{\sigma'_t\}$ preserves the $*$-operation in $\mathcal{M}$ (they are automorphisms after all), and this translates to $U(t)\xi^\sharp = (U(t)\xi)^\sharp$ for each $\xi \in \mathfrak{A}$. By density we then obtain

$$U(t)\mathfrak{D}^\sharp = \mathfrak{D}^\sharp, \qquad (U(t)\xi)^\sharp = U(t)\xi^\sharp, \qquad \xi \in \mathfrak{D}^\sharp.$$

Therefore, for each $\xi \in \mathfrak{D}^\sharp$ we have

$$\left\|\Delta^{\frac{1}{2}}\xi\right\| = \|S\xi\| = \|U(t)S\xi\| = \|SU(t)\xi\| = \left\|\Delta^{\frac{1}{2}}U(t)\xi\right\|,$$

and for $\xi, \eta \in \mathfrak{D}^\sharp$ we get

$$\begin{aligned} (\Delta^{\frac{1}{2}}\xi \mid \Delta^{\frac{1}{2}}\eta) &= (J\Delta^{\frac{1}{2}}\eta \mid J\Delta^{\frac{1}{2}}\xi) = (\eta^\sharp \mid \xi^\sharp) = (U(t)\eta^\sharp \mid U(t)\xi^\sharp) \\ &= ((U(t)\eta)^\sharp \mid (U(t)\xi)^\sharp) = (SU(t)\eta \mid SU(t)\xi) = (\Delta^{\frac{1}{2}}U(t)\xi \mid \Delta^{\frac{1}{2}}U(t)\eta) \end{aligned}$$

or

$$(\Delta\xi \mid \eta) = (U(-t)\Delta U(t)\xi \mid \eta), \qquad \xi, \eta \in \mathfrak{D}^\sharp.$$

Hence $\Delta = U(t)\Delta U(-t)$ for every $t \in \mathbb{R}$, implying that the spectral projections of Δ and $\{U(t)\}$ commute so that we can conclude

$$\Delta^{\frac{1}{2}}U(t) = U(t)\Delta^{\frac{1}{2}}, \qquad t \in \mathbb{R}.$$

Since we also know $U(t)$ commutes with $S = J\Delta^{\frac{1}{2}}$, we also have that

$$JU(t) = U(t)J, \qquad t \in \mathbb{R}.$$

Fix $\xi, \eta \in \mathfrak{D}^\sharp$ and take $\{\xi_n\}, \{\eta_n\} \subset \mathfrak{A}$ converging to ξ and η, respectively, in the $\|\cdot\|_\sharp$-norm. For each $n \in \mathbb{N}$ let $F_n := F_{\pi_l(\eta_n)^*,\pi_l(\xi_n)}$ from the modular condition of $\{\sigma'_t\}$. Then we have

$$\begin{aligned} F_n(t) &= \varphi\left(\sigma'_t(\pi_l(\eta_n)^*)\pi_l(\xi_n)\right) = (\xi_n \mid U(t)\eta_n); \\ F_n(t+\imath) &= \varphi\left(\pi_l(\xi_n)\sigma'_t(\pi_l(\eta_n)^*)\right) = (U(t)\eta_n^\sharp \mid \xi_n^\sharp). \end{aligned}$$

From the assumed convergence of $\{\xi_n\}$ and $\{\eta_n\}$ we then know that $\{F_n(t)\}$ and $\{F_n(t+i)\}$ converge uniformly in t to the functions $(\xi \mid U(t)\eta)$ and $(U(t)\eta^\sharp \mid \xi^\sharp)$ respectively. Hence from the Phragmén-Lindelöf Theorem yields the uniform convergence of $\{F_n\}$ to $F := F_{\pi_l(\eta)^*,\pi_l(\xi)}$ on $\overline{\mathbb{D}}$. Thus F is continuous, bounded on $\overline{\mathbb{D}}$, and holomorphic on $\mathbb{D}$ with boundary values:

$$F(t) = (\xi \mid U(t)\eta), \qquad F_{\pi_l(\eta)^*,\pi_l(\xi)}(t+i)(U(t)\eta^\sharp \mid \eta^\sharp).$$

Since $U(t)$ commutes with $\Delta^{\frac{1}{2}}$ and J, the second boundary value above becomes

$$F(t+\imath) = \left(\Delta^{\frac{1}{2}}\xi \mid \Delta^{\frac{1}{2}}U(t)\eta\right). \tag{6}$$

Let $K = \int_{\mathbb{R}} \lambda \, dE(\lambda)$ be the spectral decomposition of K, then

$$U(t) = \int_{\mathbb{R}} e^{\imath\lambda t} \, dE(\lambda).$$

Let $E_n := E([-n, n])$. Since $\{E(\lambda)\}$ and Δ commute, we have $E_n\mathfrak{D}^\sharp \subset \mathfrak{D}^\sharp$, and $\mathfrak{D}_0 := \bigcup_{n=1}^\infty E_n\mathfrak{D}^\sharp$ is a core for $\Delta^{\frac{1}{2}}$ because $\lim_{n\to\infty}(1+\Delta^{\frac{1}{2}})E_n\xi = \lim_{n\to\infty} E_n(1+\Delta^{\frac{1}{2}})\xi$ for every $\xi \in \mathfrak{D}^\sharp$. Now, if $\xi \in \mathfrak{D}^\sharp$ and $\eta \in \mathfrak{D}_0$, then for sufficiently large n we have

$$F(t+\imath) = \int_{-n}^{n} e^{-\imath\lambda(t+\imath)}(\xi \mid dE(\lambda)\eta) = \int_{-n}^{n} e^{\lambda}e^{-\imath\lambda t}(\xi \mid dE(\lambda)\eta) = (\xi \mid HU(t)\eta).$$

Comparing this to our previous computation in 6 we have

$$\left(\Delta^{\frac{1}{2}}\xi \mid \Delta^{\frac{1}{2}}U(t)\eta\right) = (\xi \mid HU(t)\eta), \qquad \xi \in \mathfrak{D}^\sharp,\ \eta \in \mathfrak{D}_0.$$

In particular, for $t = 0$ we have $(\Delta^{\frac{1}{2}}\xi \mid \Delta^{\frac{1}{2}}\eta) = (\xi \mid H\eta)$. Hence $\Delta^{\frac{1}{2}}\mathfrak{D}_0 \subset \mathfrak{D}^\sharp$ and $H\eta = \Delta\eta$ for each $\eta \in \mathfrak{D}_0$. On the other hand, $(1+H)\mathfrak{D}_0 = \bigcup_{n=1}(1+H)E_n\mathfrak{D}^\sharp$, and $(1+H)E_n\mathfrak{D}^\sharp$ is dense in $E_n\mathfrak{H}_\varphi$, so that $\mathfrak{D}_0$ is dense in $\mathfrak{D}(H)$ with respect to the graph norm. Similarly, $\mathfrak{D}_0$ is dense in $\mathfrak{D}(\Delta)$ with respect to the graph norm. In other words, $\mathfrak{D}_0$ is a common core for H and Δ, on which H and Δ agree. Therefore $H = \Delta$ as needed. □

Definition 5.3. The one parameter automorphism group $\{\sigma_t\}$ of $\mathcal{M}$ given by a weight φ is called the **modular automorphism group** associated with φ and denoted by $\{\sigma_t^\varphi\}$.

Henceforth we let $\mathcal{A}(\mathbb{D})$ denote the set of bounded continuous functions on the closed horiztonal strip $\overline{\mathbb{D}}$ which are holomorphic on $\mathbb{D}$. The function $F_{x,y} \in \mathcal{A}(\mathbb{D})$ in Definition 5.1 is called the two point function of x and y.

Corollary 5.4. *Let π be an isomorphism of a von Neumann algebra $\mathcal{M}$ onto another $\mathfrak{N}$. If ψ is a faithful semi-finite normal weight on $\mathbb{N}$, then*

$$\sigma_t^{\psi\circ\pi} = \pi^{-1} \circ \sigma_t^\psi \circ \pi, \qquad t \in \mathbb{R}.$$

Proof. This follows from the uniqueness of modular automorphism group established in Theorem **??**. □

We now proceed onto the topic of the "centralizer of a weight." Starting now, we only assume weights to be semi-finite and normal (not necessarily faithful).

Fix a faithful weight φ on a von Neumann algebra $\mathcal{M}$.

Definition 5.5. Let

$$\mathcal{M}_\varphi := \left\{x \in \mathcal{M} \colon \sigma_t^\varphi(x) = x,\ t \in \mathbb{R}\right\}.$$

Then $\mathcal{M}_\varphi$ is a von Neumann subalgebra of $\mathcal{M}$ called the **centralizer** of φ. We say that $x \in \mathcal{M}$ and φ **commute** if $x \in \mathcal{M}_\varphi$.

Our initial goal is to establish criteria in terms of φ for an $x \in \mathcal{M}$ to commute with φ. The terms defined in the following have been used above, but only with the implicit algebraic conditions. Here we add an analytic condition and this is assumed henceforth.

Definition 5.6. We mean by a **one parameter automorphism group** $\{\sigma_t\}$ of $\mathcal{M}$ a homomorphism $\sigma\colon t \in \mathbb{R} \mapsto \sigma_t \in \mathrm{Aut}(\mathcal{M})$ from the additive group $\mathbb{R}$ into the group $\mathrm{Aut}(\mathcal{M})$ of automorphisms of $\mathcal{M}$ with the continuity requirement that for every $x \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$ the function $t \mapsto \omega(\sigma_t(x)) \in \mathbb{C}$ is continuous. An element $x \in \mathcal{M}$ is said to be **entire** if the function $t \mapsto \sigma_t(x) \in \mathcal{M}$ can be exteneded to an $\mathcal{M}$-valued entire function over $\mathbb{C}$. In this case, the value at $z \in \mathbb{C}$ will be denoted by $\sigma_z(x)$. We denote by $\mathcal{M}_a^\sigma$ the set of all entire elements of $\mathcal{M}$.

Lemma 5.7. *Let $\{\sigma_t\}$ be a one parameter automorphism group of $\mathcal{M}$.*

(i) *$\mathcal{M}_a^\sigma$ is a σ-weakly dense $*$-subalgebra of $\mathcal{M}$;*
(ii) *If $x, y \in \mathcal{M}_a^\sigma$ and $\alpha, \beta \in \mathbb{C}$, then*

$$\begin{aligned}\sigma_\alpha(xy) &= \sigma_\alpha(x)\sigma_\alpha(y),\\ \sigma_{\alpha+\beta}(x) &= \sigma_\alpha(\sigma_\beta(x)) = \sigma_\beta(\sigma_\alpha(x)),\\ \sigma_{\overline{\alpha}}(x) &= \sigma_\alpha(x^*)^*.\end{aligned}$$

Proof.

(i): For each $x \in \mathcal{M}_a^\sigma$, we set

$$\sigma_a'(x) = \left.\frac{d}{d\beta}\sigma_\beta(x)\right|_{\beta=\alpha} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(\sigma_{\alpha+\epsilon}(x) - \sigma_\alpha(x)).$$

For each $x, y \in \mathcal{M}_a^\sigma$, we have

$$\begin{aligned}\frac{1}{\epsilon}(\sigma_{\alpha+\epsilon}(x)\sigma_{\alpha+\epsilon}(y) - \sigma_\alpha(x)\sigma_\alpha(y)) &= \frac{1}{\epsilon}\left([\sigma_{\alpha+\epsilon}(x) - \sigma_\alpha(x)]\sigma_{\alpha+\epsilon}(y) + \sigma_\alpha(x)[\sigma_{\alpha+\epsilon}(y) - \sigma_\alpha(y)]\right)\\ &\xrightarrow{\epsilon\to 0} \sigma_\alpha'(x)\sigma_\alpha(y) + \sigma_\alpha(x)\sigma_\alpha'(y).\end{aligned}$$

Thus $\alpha \in \mathbb{C} \mapsto \sigma_\alpha(x)\sigma_\alpha(y) \in \mathcal{M}$ is entire. Hence xy is entire, $xy \in \mathcal{M}_a^\sigma$, and $\sigma_\alpha(xy)$ is an extension of $\sigma_t(xy)$. On the other hand, since $\sigma_t(x)\sigma_t(y) = \sigma_t(xy)$ we know by the uniqueness of the extension that $\sigma_\alpha(xy) = \sigma_\alpha(x)\sigma_\alpha(y)$. Thus $\mathcal{M}_a^\sigma$ is a subalgebra of $\mathcal{M}$.

To see that $\mathcal{M}_a^\sigma$ is closed under taking adjoint, we note that if $\alpha \mapsto \sigma_\alpha(x)$ is entire, then $\alpha \mapsto \sigma_{\bar{\alpha}}(x)^*$ is entire. This latter function extends $t \mapsto \sigma_t(x)^* = \sigma_t(x^*)$ so we get that $x^* \in \mathcal{M}_\sigma^a$, ergo $\mathcal{M}_\sigma^\alpha$ is a $*$-subalgebra of $\mathcal{M}$ and $\sigma_{\bar{\alpha}}(x) = \sigma_\alpha(x^*)^*$.

Given $x \in \mathcal{M}_a^\sigma$ and $s \in \mathbb{R}$, we claim that $\sigma_s(x) \in \mathcal{M}_a^\varphi$. Indeed, $\sigma_t(\sigma_s(x)) = \sigma_{t+s}(x)$ is extended by the entire function $\sigma_{\alpha+s}(x)$. Also, the uniqueness of the holmorphic extension we obtain $\sigma_\alpha \circ \sigma_s = \sigma_{\alpha+s}$. This also provides us with an extension for $t \mapsto \sigma_s \circ \sigma_t(x) = \sigma_{s+t}(x)$ and hence we also obtain $\sigma_{s+\alpha} = \sigma_s \circ \sigma_\alpha$. Now, fix $\alpha \in \mathbb{C}$, then $\sigma_{\beta+\alpha}$ is an extension for $t \mapsto \sigma_t(\sigma_\alpha(x)) = \sigma_{t+\alpha}(x)$, so that $\sigma_\alpha(x) \in \mathcal{M}_a^\sigma$ and $\sigma_\beta \circ \sigma_\alpha = \sigma_{\beta+\alpha}$.

It remains to show $\mathcal{M}_a^\sigma$ is σ-weakly dense in $\mathcal{M}$. For each $x \in \mathcal{M}$, set

$$x_\gamma(\alpha) := \sqrt{\frac{\gamma}{\pi}}\int_{\mathbb{R}} e^{-\gamma(t-\alpha)^2}\sigma_t(x)\ dt, \qquad \gamma > 0,\ \alpha \in \mathbb{C}.$$

For each $\omega \in \mathcal{M}_*$, we have

$$\omega(x_\gamma(\alpha)) = \sqrt{\frac{\gamma}{\pi}}\int_{\mathbb{R}} e^{-\gamma(t-\alpha)^2}\omega(\sigma_t(x),)\ dt.$$

The boundedness of the function $t \mapsto \omega(\sigma_t(x))$ yields the analyticity of the right hand side as a function of α. Hence $x_\gamma(\cdot)$ is an $\mathcal{M}$-valued entire function which extends the function: $t \mapsto \sigma_t(x_\gamma(0)) = x_\gamma(t)$. Thus $x_\gamma := x_\gamma(0) \in \mathcal{M}_a^\sigma$. By Lemma 2.4, $\{x_\gamma\}$ converges σ-weakly to x as $\gamma \to \infty$. Hence $\mathcal{M}_a^\sigma$ is σ-weakly dense in $\mathcal{M}$. ☐

Given the modular autormorphism group $\{\sigma_t^\varphi\}$ of a faithful weight φ, we will use $\mathcal{M}_a^\varphi$ (rather than $\mathcal{M}_a^{\sigma^\varphi}$) to denote the set of entire elements. Let $\mathfrak{A}$ and $\mathfrak{A}_0$ be the left Hilbert and Tomita algebras, respectively, corresponding to φ through the constructions in Sections 4 and 2. We set

$$\mathfrak{a}_\varphi = \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^* = \pi_l(\mathfrak{A}), \qquad \mathfrak{a}_0 = \pi_l(\mathfrak{A}_0).$$

From Definition 2.1.(a), we see that $\mathfrak{a}_0 \subset \mathcal{M}_a^\varphi$ and

$$\sigma_\alpha^\varphi(\pi_l(\xi)) = \pi_l(\Delta^{i\alpha}\xi), \qquad \xi \in \mathfrak{A}_0.$$

Lemma 5.8.

(i) $\mathfrak{a}_\varphi$ *is an* $\mathcal{M}_a^\varphi$*-bimodule.*
(ii) $\mathfrak{m}_\varphi$ *is an* $\mathcal{M}_a^\varphi$*-bimodule.*
(iii) $\mathfrak{a}_0$ *is an ideal of* $\mathcal{M}_a^\varphi$*.*

Proof.

(i): Since $\mathfrak{a}_\varphi$ and $\mathcal{M}_a^\varphi$ are both $*$-algebras, it suffices to prove $\mathfrak{a}_\varphi$ is a left $\mathcal{M}_a^\varphi$-module. For this, it is also sufficient to prove $\mathcal{M}_a^\varphi\mathfrak{a}_\varphi \subset \mathfrak{n}_\varphi^*$, since $\mathfrak{n}_\varphi$ is a left ideal and $\mathfrak{a}_\varphi \subset \mathfrak{n}_\varphi$. Given $a \in \mathcal{M}_a^\varphi$ and $x \in \mathfrak{a}_\varphi$, having $ax \in \mathfrak{n}_\varphi^*$ is true iff $ax = b^*$ for some $b \in \mathfrak{n}_\varphi$ iff $(ax)^* = b \in \mathfrak{n}_\varphi$ iff $\eta_\varphi(ax) \in \mathfrak{D}^\sharp = \mathfrak{D}(\Delta^{1/2})$ (since $S = J\Delta^{1/2}$). Now,

$$\Delta^{it}\eta_\varphi(ax) = \eta_\varphi(\sigma_t^\varphi(ax)) = \sigma_t^\varphi(a)\eta_\varphi(\sigma_t^\varphi(x)) = \sigma_t^\varphi(a)\Delta^{it}\eta_\varphi(x).$$

We know $\eta_\varphi(x) \in \mathfrak{D}(\Delta^{1/2})$ because $x \in \mathfrak{a}_\varphi$, and thus applying Lemma 2.3 (to $H = \Delta$), we see that $t \mapsto \Delta^{it}\eta_\varphi(x)$ can be extended to an $\mathfrak{h}$-valued holmorphic function on the horizontal strip bounded by $\mathbb{R}$ and $\mathbb{R}-i\frac{1}{2}$, continuous on the closue. But $\varphi_t^\varphi(a)$ extends to an $\mathcal{M}$-valued entire function and so their composition, which is equivalent to $\Delta^{it}\eta_\varphi(ax)$ by the above computation, is holomorphic on the strip and continuous on its closure. Hence Lemma 2.3 applied again yields that $\eta_\varphi(ax)\in\mathfrak{D}(\Delta^{1/2})$.

(ii): Let $a\in\mathcal{M}_a^\varphi$ and $x\in\mathfrak{m}_\varphi^+=\mathfrak{p}_\varphi$. Then $x^{\frac{1}{2}}\in\mathfrak{n}_\varphi\cap\mathfrak{n}_\varphi^*$ since it is self-adjoint and $(x^{\frac{1}{2}})x^{\frac{1}{2}}=x\in\mathfrak{p}_\varphi$. So by part (i), $ax^{\frac{1}{2}}\in\mathfrak{n}_\varphi\cap\mathfrak{n}_\varphi^*$ and hence $ax=(ax^{\frac{1}{2}})x^{\frac{1}{2}}\in\mathfrak{m}_\varphi=\mathfrak{n}_\varphi^*\mathfrak{n}_\varphi$. As $\mathfrak{m}_\varphi$ is spanned by its positive elements we conclude that it is a left $\mathcal{M}_a^\varphi$-module. That it is also a right module follows from $\mathfrak{m}_\varphi a=(a^*\mathfrak{m}_\varphi)^*$.

(iii): As $\mathfrak{a}_0$ is a $*$-algebra it suffices to show it is a left ideal. Let $a\in\mathcal{M}_a^\varphi$ and $x\in\mathfrak{a}_0$. Then $x=\pi_l(\xi)$ for some $\xi\in\mathfrak{A}_0$ and $ax=\pi_l(a\xi)$. Since $\Delta^{it}a\xi=\sigma_t^\varphi(a)\Delta^{it}\xi$ extends to an $\mathfrak{H}$ valued entire function (because $\Delta^{it}\xi$ does and $\sigma_t^\varphi(a)$ extends to an $\mathcal{M}$ valued entire function) $a\xi\in\mathfrak{A}_0=\bigcap_{n\in\mathbb{Z}}\mathfrak{D}(\Delta^n)$ by Lemma 2.3. Hence $ax=\pi_l(a\xi)\in\mathfrak{a}_0$. □

Lemma 5.9.

(i) *If $a\in\mathcal{M}$ is a multiplier of $\mathfrak{m}_\varphi$ in the sense that $a\mathfrak{m}_\varphi\subset\mathfrak{m}_\varphi$ and $\mathfrak{m}_\varphi a\subset\mathfrak{m}_\varphi$, then for each $x,y\in\mathfrak{a}_0$ there exists an entire $F\in\mathcal{A}(\mathbb{D})$ such that*

$$F(t)=\varphi(\sigma_t^\varphi(a)xy^*),\qquad F(t+i)=\varphi(xy^*\sigma_t^\varphi(a)).\tag{7}$$

(ii) *If $a\in\mathcal{M}_a^\varphi$ and $z\in\mathfrak{m}_\varphi$, then the function F_z defined by $\varphi(\sigma_\alpha^\varphi(a)z)$ is entire and bounded on $\mathbb{D}$, and further satisfies the condition:*

$$F_z(t)=\varphi(\sigma_t^\varphi(a)z),\qquad F_z(t+i)=\varphi(z\sigma_t^\varphi(a)).\tag{8}$$

Proof.

(i): Set

$$F(\alpha)=(a\Delta^{-i\alpha}\eta_\varphi(x)\mid\Delta^{-i\bar{\alpha}+1}\eta_\varphi(y)),\qquad\alpha\in\mathbb{C}.$$

By the assumption on x and y, F is an entire function and belongs to $\mathfrak{A}(\mathbb{D})$. Now, $x\in\mathfrak{a}_0\subset\mathfrak{m}_\varphi$ and since σ_t^φ leaves $\mathfrak{m}_\varphi$ invariant this means $\sigma_{-t}^\varphi(x)\in\mathfrak{m}_\varphi$. Consequently $a\sigma_{-t}^\varphi(x)\in\mathfrak{m}_\varphi$ by assumption on a, and so $\sigma_t^\varphi(a)x\in\sigma_t^\varphi(\mathfrak{m}_\varphi)=\mathfrak{m}_\varphi$. Hence $\sigma_t^\varphi(a)\eta_\varphi(x)=\eta_\varphi(\sigma_t^\varphi(a)x)$. Using $\mathfrak{m}_\varphi a\subset\mathfrak{m}_\varphi$ and a similar argument we can also show $\sigma_t^\varphi(a^*)y\in\mathfrak{m}_\varphi$, so that $\sigma_t^\varphi(a^*)\eta_\varphi(y)=\eta_\varphi(\sigma_t^\varphi(a^*)y)$. Now, we compute:

$$\begin{aligned}F(t)&=(a\Delta^{-it}\eta_\varphi(x)\mid\Delta^{-it+1}\eta_\varphi(y))=(\sigma_t^\varphi(a)\eta_\varphi(x)\mid\eta_\varphi(y))\\&=(\Delta^{\frac{1}{2}}\eta_\varphi(\sigma_t^\varphi(a)x)\mid\Delta^{\frac{1}{2}}\eta_\varphi(y))=(S\eta_\varphi(y)\mid S\eta_\varphi(\sigma_t^\varphi(a)x))=\varphi\left(\sigma_t^\varphi(a)xy^*\right);\\F(t+i)&=(a\Delta^{-it+1}\eta_\varphi(x)\mid\Delta^{-it}\eta_\varphi(y))=(\Delta^{\frac{1}{2}}\eta_\varphi(x)\mid\Delta^{\frac{1}{2}}\sigma_t^\varphi(a^*)\eta_\varphi(y))\\&=(S\eta_\varphi(\sigma_t^\varphi(a^*)y)\mid S\eta_\varphi(x))=\varphi\left(xy^*\sigma_t^\varphi(a)\right).\end{aligned}$$

(ii): It suffices to assume $z=xy^*$, $x,y\in\mathfrak{a}_\varphi$ since $\mathfrak{m}_\varphi$ spans $\mathfrak{m}_\varphi$ linearly and $z\mapsto\varphi(\sigma_\alpha^\varphi(a)z)=F_z(\alpha)$ is linear in z. Now, the previous lemma implies $\eta_\varphi(bx)\in\mathfrak{A}\subset\mathfrak{D}(\Delta^{1/2})$ for $b\in\mathcal{M}_a^\varphi$ and

$$\sigma_\alpha^\varphi(b)\Delta^{i\alpha}\eta_\varphi(x)=\Delta^{i\alpha}\eta_\varphi(bx),\qquad\alpha\in\overline{\mathbb{D}}_{\frac{1}{2}}.$$

In particular, applying this to $b=\sigma_\beta^\varphi(a)$ and $\alpha=-\frac{i}{2}$,

$$\Delta^{\frac{1}{2}}\sigma_\beta^\varphi(a)\eta_\varphi(x)=\sigma_{\beta-\frac{i}{2}}^\varphi(a)\Delta^{\frac{1}{2}}\eta_\varphi(x),\qquad\beta\in\mathbb{C}.$$

We compute:

$$\begin{aligned}F_z(\alpha)&=\varphi(\sigma_\alpha^\varphi(a)xy^*)=(S\eta_\varphi(y)\mid S\eta_\varphi(\sigma_\alpha^\varphi(a)x))\\&=(\Delta^{\frac{1}{2}}\sigma_\alpha^\varphi(a)\eta_\varphi(x)\mid\Delta^{\frac{1}{2}}\eta_\varphi(y))=\left(\sigma_{\alpha-\frac{i}{2}}^\varphi(a)\Delta^{\frac{1}{2}}\eta_\varphi(x)\mid\Delta^{\frac{1}{2}}\eta_\varphi(y)\right).\end{aligned}$$

Hence F_z is entire and bounded on $\bar{\mathbb{D}}_{1/2}$. Also,

$$\begin{aligned}F_z(t+i)&=\left(\sigma_{t+\frac{i}{2}}^\varphi(a)\Delta^{\frac{1}{2}}\eta_\varphi(x)\mid\Delta^{\frac{1}{2}}\eta_\varphi(y)\right)=\left(\Delta^{\frac{1}{2}}\eta_\varphi(x)\mid\sigma_{t-\frac{i}{2}}^\varphi(a^*)\Delta^{\frac{1}{2}}\eta_\varphi(y)\right)\\&=\left(\Delta^{\frac{1}{2}}\eta_\varphi(x)\mid\Delta^{\frac{1}{2}}\sigma_t^\varphi(a^*)\eta_\varphi(y)\right)=(S\eta_\varphi(\sigma_t^\varphi(a^*)y)\mid S\eta_\varphi(x))\\&=\varphi\left(xy^*\sigma_t^\varphi(a)\right)=\varphi(z\sigma_t^\varphi(a)).\end{aligned}$$

□

We can know characterize the centralizer of a weight.

Theorem 5.10. *Let φ be a faithful semi-finite normal weight on a von Neumann algebra $\mathcal{M}$. A necessary and sufficient condition for an element $a \in \mathcal{M}$ to belong to the centralizer $\mathcal{M}_\varphi$ of φ is that*

(i) *a is a multiplier of $\mathfrak{m}_\varphi$, i.e., $a\mathfrak{m}_\varphi \subset \mathfrak{m}_\varphi$ and $\mathfrak{m}_\varphi a \subset \mathfrak{m}_\varphi$;*
(ii) *$\varphi(az) = \varphi(za)$, $z \in \mathfrak{m}_\varphi$.*

Proof. First suppose $a \in \mathcal{M}_\varphi$. Then $t \mapsto \sigma_t^\varphi(a) = a$ is extended by the constant entire function $\alpha \mapsto a$. Hence $a \in \mathcal{M}_a^\varphi$. Since $\mathfrak{m}_\varphi$ is an $\mathcal{M}_a^\varphi$-bimodule, a is a multiplier of $\mathfrak{m}_\varphi$. For $z \in \mathfrak{m}_\varphi$, part (ii) of the previous lemma gives rise to an entire $F_z \in \mathcal{A}(\mathbb{D})$ satisfying (8). But $F_z(t) = \varphi(az)$ for all $t \in \mathbb{R}$ and hence is constant everywhere. In particular $\varphi(az) = F_z(t) = F_z(t+i) = \varphi(za)$.

Conversely, suppose $a \in \mathcal{M}$ satisfies (i) and (ii) above. Then for any $x, y \in \mathfrak{a}_0$ we can produce an entire $F \in \mathcal{A}(\mathbb{D})$ satisfying (7). Then

$$F(t) = \varphi \circ \sigma_t^\varphi(a\sigma_{-t}^\varphi(xy^*)) = \varphi(a\sigma_{-t}^\varphi(xy^*)) = \varphi(\sigma_{-t}^\varphi(xy^*)a) = \varphi(xy^*\sigma_t^\varphi(a)) = F(t+i),$$

so F has period i. But F is bounded on $\bar{D}$ and entire, hence it is constant by the Liouville theorem. From the construction of F we obtain

$$\left(a\Delta^{-it}\eta_\varphi(x) \mid \Delta^{-it+1}\eta_\varphi(y)\right) = (a\eta_\varphi(x) \mid \Delta\eta_\varphi(y)), \qquad x, y \in \mathfrak{a}_0,$$

so that

$$((\sigma_t^\varphi(a) - a)\,\eta_\varphi(x) \mid \Delta\eta_\varphi(y)) = 0, \qquad x, y \in \mathfrak{a}_0.$$

The density of $\eta_\varphi(\mathfrak{a}_0) = \mathfrak{A}_0$ and $\Delta\mathfrak{A}_0 = \mathfrak{A}_0$ in $\mathfrak{H}_\varphi$ imply $\sigma_t^\varphi(a) = a$ for all $t \in \mathbb{R}$. Hence $a \in \mathcal{M}_\varphi$. $\square$

We next explore how perturbing φ by a positive self-adjoint operator affiliated with $\mathcal{M}_\varphi$ affects the modular automorphism group. Given a positive self-adjoint operator h on a Hilbert space $\mathfrak{H}$, we define

$$h_\epsilon = h(1+\epsilon h)^{-1}, \qquad \epsilon > 0.$$

Then h_ϵ is bounded and self-adjoint. For two such operator h and k on $\mathfrak{H}$ we write $h \leq k$ if there exists $\epsilon > 0$ such that $h_\epsilon \leq k_\epsilon$. This is equivalent to the fact that $h_\epsilon \leq k_\epsilon$ for all $\epsilon > 0$ via the functional calulus ($f_\epsilon(x) := \frac{x}{1+\epsilon x}$ is monotone increasing for all $\epsilon > 0$). But then this is in turn equivalent to

$$\mathfrak{D}(h^{\frac{1}{2}}) \supset \mathfrak{D}(k^{\frac{1}{2}}) \qquad \text{and} \qquad \|h^{\frac{1}{2}}\xi\| \leq \|k^{\frac{1}{2}}\xi\|, \qquad \xi \in \mathfrak{D}(k^{\frac{1}{2}}).$$

Fix a faithful weight φ on $\mathcal{M}$.

Lemma 5.11. *For each $h \in \mathcal{M}_\varphi^+$, if we set*

$$\varphi_h(x) = \varphi(h^{\frac{1}{2}}xh^{\frac{1}{2}}), \qquad x \in \mathcal{M}_+,$$

then φ_h is a weight on $\mathcal{M}$. The map: $h \mapsto \varphi_h$ is a monotone increasing [affine??] map.

Proof. By Theorem 5.10, h is a multiplier of $\mathfrak{m}_\varphi$ so that φ_h takes finite values on $\mathfrak{m}_\varphi^+$ and hence is semi-finite. The normality of φ_h follows from that of φ.

If $h, k \in \mathcal{M}_\varphi^+$, then by Lemma 3.6 there exists $u, v \in \mathcal{M}_\varphi$ such taht $\|u\| \leq 1$, $\|v\| \leq 1$ and

$$h^{\frac{1}{2}} = u(h+k)^{\frac{1}{2}}, \qquad k^{\frac{1}{2}} = v(h+k)^{\frac{1}{2}}, \qquad u^*u + v^*v = s(h+k),$$

where $s(h+k)$ is the range projection of $h+k$. Suppose that $\varphi_{(h+k)}(x) < \infty$ for some $x \in \mathcal{M}_+$. Then

$$y := (h+k)^{\frac{1}{2}}x(h+k)^{\frac{1}{2}} \in \mathfrak{m}_\varphi^+.$$

Then $uyu^*, vyv^* \in \mathfrak{m}_\varphi$ since $u, v \in \mathcal{M}_\varphi$ are multipliers of $\mathfrak{m}_\varphi$. But then condition (ii) in Theorem 5.10 implies

$$\varphi_h(x) + \varphi_k(x) = \varphi(uyu^* + vyv^*) = \varphi(u^*uy + v^*vy) = \varphi(s(h+k)y) = \varphi(y) = \varphi_{(h+k)}(x).$$

Now suppose $\varphi_h(x), \varphi_k(h) < \infty$. Then

$$\begin{aligned}(h+k)^{\frac{1}{2}}x(h+k)^{\frac{1}{2}} &= \lim_{\epsilon \to 0}(h+k+\epsilon)^{-\frac{1}{2}}(h+k)x(h+k)(h+k+\epsilon)^{-\frac{1}{2}} \\ &\leq 2\lim_{\epsilon \to 0}(h+k+\epsilon)^{-\frac{1}{2}}(hxh+kxk)(h+k+\epsilon)^{-\frac{1}{2}} \\ &= 2\left(u^*h^{\frac{1}{2}}xh^{\frac{1}{2}}u + v^*k^{\frac{1}{2}}xk^{\frac{1}{2}}v\right),\end{aligned}$$

so that by Lemma 5.8 $\varphi_{(h+k)}(x) < \infty$. But then the preceding arugment implies $\varphi_{(h+k)}(x) = \varphi_h(x) + \varphi_k(x)$. Thus $\varphi_h + \varphi_k = \varphi_{(h+k)}$.

If $h \leq k$, then $k = h + (k - h)$, so we have

$$\varphi_k = \varphi_h + \varphi_{(k-h)} \geq \varphi_h.$$

□

Lemma 5.12. *Let h be a positive self-adjoint operator affiliated with the centralizer $\mathcal{M}_\varphi$ of φ. Then the right hand side of the following:*

$$\varphi_h(x) = \lim_{\epsilon \to 0} \varphi\left(h_\epsilon^{\frac{1}{2}} x h_\epsilon^{\frac{1}{2}}\right), \qquad x \in \mathcal{M}_+, \tag{9}$$

converges in $[0, \infty]$ and gives a weight φ_h on $\mathcal{M}$. A necessary and sufficient condition for φ_h to be faithful is that h is non-singular, i.e. the range of h is dense in $\mathfrak{H}_\varphi$ or equivalently $h\xi \neq 0$ for every non-zero $\xi \in \mathfrak{D}(h)$.

Proof. The inequality, $0 < \epsilon < \delta$, implies $h_\delta \leq h_\epsilon$, so that $\{\varphi_{h_\epsilon} : \epsilon > 0\}$ is monotone increasing by the previous lemma in the sense that $\epsilon \searrow 0$ gives $\varphi_{h_\epsilon} \nearrow \varphi_h$. Hence $\varphi = \sup \varphi_{h_\epsilon} = \lim_{\epsilon \searrow 0} \varphi_{h_\epsilon}$ makes sense and is linear on $\mathcal{M}_+$ by the linearity of each φ_{h_ϵ}. The normality of φ_h follows from that of each φ_{h_ϵ}.

Now, we prove the semi-finiteness of φ_h. Let e_n be the spectral projection of h corresponding to the interval $[0, n]$. By Lemma 5.8, $e_n \mathfrak{m}_\varphi e_n$ is σ-weakly dense in $e_n \mathcal{M} e_n$, so that $\bigcup_{n=1}^\infty e_n \mathfrak{m}_\varphi e_n$ is σ-weakly dense in $\mathcal{M}$ [I THINK THIS PROCEDES BY: since h is affiliated with $\mathcal{M}_\varphi$, $e_n \in \mathcal{M}_\varphi \subset \mathcal{M}_a^\varphi$ whence $e_n \mathfrak{m}_\varphi e_n \subset \mathfrak{m}_\varphi$ by the lemma mentioned. The σ-weak density comes from the fact that since φ is semi-finite, $\mathfrak{m}_\varphi^+ = \mathfrak{p}_\varphi$ generates $\mathcal{M}$.] But φ_h takes finite values on $e_n \mathfrak{m}_\varphi e_n$ because $he_n \in \mathcal{M}_\varphi^+$, which yields the semi-finiteness of φ_h. If $e = s(h)$ is the range projection of h, then $\varphi_h(1 - e) = 0$, so that the faithfulness of φ_h is equivqalent to $e = 1$. □

Lemma 5.13. *If $h \in \mathcal{M}_\varphi^+$ is invertible, and if $x \in \mathfrak{a}_0$ and $y \in \mathfrak{a}_\varphi$, then the entire function F:*

$$F(\alpha) = \left(h^{i\alpha+1} \Delta^{i\alpha+1} \eta_\varphi(x) \mid S h^{-i\alpha} \eta_\varphi(y)\right)$$

belongs to $\mathcal{A}(\mathbb{D})$ and satisfies the boundary conditions:

$$F(t) = \varphi\left(h h^{it} \sigma_t^\varphi(x) h^{-it} y\right),$$
$$F(t+i) = \varphi\left(h y h^{it} \sigma_t^\varphi(x) h^{-it}\right).$$

Proof. That F is entire is clear from $x \in \mathfrak{a}_0$. We compute

$$\begin{aligned}
F(t) &= (h^{it+1} \Delta^{it+1} \eta_\varphi(x) \mid S h^{-it} \eta_\varphi(y)) = (h^{it+1} \Delta \eta_\varphi(\sigma_t^\varphi(x)) \mid \eta_\varphi(y^* h^{it})) \\
&= (\Delta^{\frac{1}{2}} \eta_\varphi(\sigma_t^\varphi(x)) \mid \Delta^{1/2} \eta_\varphi(h^{-it+1} y^* h^{it})) = (S \eta_\varphi(h^{-it+1} y^* h^{it}) \mid S \eta_\varphi(\sigma_t^\varphi(x))) \\
&= (\eta_\varphi(h^{-it} y h^{it+1}) \mid \eta_\varphi(\sigma_t^\varphi(x)^*)) = \varphi(\sigma_t^\varphi(x) h^{-it} y h^{it+1}) = \varphi(h h^{it} \sigma_t^\varphi(x) h^{-it} y); \\
F(t+i) &= (h^{it} \Delta^{it} \eta_\varphi(x) \mid S h^{-it+1} \eta_\varphi(y)) = (h^{it} \eta_\varphi(\sigma_t^\varphi(x)) \mid S h^{-it+1} \eta_\varphi(y)) \\
&= \varphi(h^{-it+1} y h^{it} \sigma_t^\varphi(x)) = \varphi(h y h^{it} \sigma_\varphi(x) h^{-it})
\end{aligned}$$

□

Lemma 5.14. *If $h \in \mathcal{M}_\varphi^+$ is invertible, then the modular automorphism group $\{\sigma_t^\psi\}$ of $\psi := \varphi_h$ is given by the following:*

$$\sigma_t^\psi(x) = h^{it} \sigma_t^\varphi(x) h^{-it}, \qquad x \in \mathcal{M}, \ t \in \mathbb{R}.$$

Proof. Since h is a multiplier of $\mathfrak{m}_\varphi$ we have

$$\mathfrak{m}_\psi = h^{-\frac{1}{2}} \mathfrak{m}_\varphi h^{-\frac{1}{2}} \subset \mathfrak{m}_\varphi, \qquad \text{and} \qquad \mathfrak{m}_\varphi = h^{-\frac{1}{2}} h^{\frac{1}{2}} \mathfrak{m}_\varphi h^{\frac{1}{2}} h^{-\frac{1}{2}} \subset \mathfrak{m}_\varphi,$$

so that $\mathfrak{m}_\varphi = \mathfrak{m}_\psi$ and $\mathfrak{n}_\varphi = \mathfrak{n}_\psi$. Let $x, y \in \mathfrak{a}_\psi$ and let $\{x_n\} \subset \mathfrak{a}_0$ be a sequence such that $\lim_n \|\eta_\varphi(x) - \eta_\varphi(x_n)\|_\sharp = 0$. From the previous Lemma we have a sequence of entire functions $\{F_n\} \subset \mathcal{A}(\mathbb{D})$ such that

$$F_n(t) = \varphi(h h^{it} \sigma_t^\varphi(x_n) h^{-it} y) = (\eta_\varphi(h^{-it} h^{it} h) \mid \Delta^{it} \eta_\varphi(x_n^*)),$$
$$F_n(t+i) = \varphi(h y h^{it} \sigma_t^\varphi(x_n) h^{it}) = (\Delta^{it} \eta_\varphi(x_n) \mid \eta_\varphi(h^{-it} y^* h^{it} h)).$$

But since the convergence

$$\lim_{n \to \infty} \|\Delta^{it}(\eta_\varphi(x) - \eta_\varphi(x))\|_\sharp = 0$$

is uniform in $t \in \mathbb{R}$, so the sequence $\{F_n\}$ converges uniformly on the boundaries $\mathbb{R}$ and $\mathbb{R}+i$ of $\mathbb{D}$. So the Phragmén-Lindelöf theorem implies that $\{F_n\}$ converges to an $F \in \mathcal{A}(\mathbb{D})$ whose boundary values are given by:

$$F(t) = \lim_{n\to\infty} F_n(t) = (\eta_\varphi(h^{-it}yh^{it}h) \mid \eta_\varphi(\sigma_t^\varphi(x^*))) = \varphi\left(h^{\frac{1}{2}}h^{it}\sigma_t^\varphi(x)h^{-it}yh^{\frac{1}{2}}\right) = \psi\left(h^{it}\sigma_t^\varphi(x)h^{-it}y\right);$$

$$F(t+i) = \lim_{n\to\infty} F_n(t+i) = (\eta_\varphi(\sigma_t^\varphi(x)) \mid \eta_\varphi(h^{-it}y^*h^{it}h)) = \varphi\left(h^{\frac{1}{2}}yh^{it}\sigma_t^\varphi(x)h^{-it}h^{\frac{1}{2}}\right) = \psi\left(yh^{it}\sigma_t^\varphi(x)h^{-it}\right).$$

Hence $h^{it}\sigma_t^\varphi(x)h^{-it}$ satisfies the modular condition for ψ. We also note that

$$\psi(h^{it}\sigma_t^\varphi(x)h^{-it}) = \varphi\left(h^{\frac{1}{2}}h^{it}\sigma_t^\varphi(x)h^{-it}h^{\frac{1}{2}}\right) = \varphi(h\sigma_t^\varphi(x)) = \varphi(\sigma_t^\varphi(hx)) = \varphi(hx) = \psi(x).$$

So by the uniqueness of the modular automorphism group we obtain $\sigma_t^\psi(x) = h^{it}\sigma_t^\varphi(x)h^{-it}$. □

We expand this result to positive self-adjoint operators affiliated with $\mathcal{M}_\varphi$:

Theorem 5.15. *Let φ be a faithful semi-finite normal weight on a von Neumann algebra $\mathcal{M}$. If h is a non-singular positive self-adjoint operator affiliated with the centralizer $\mathcal{M}_\varphi$ of φ, then the modular automorphism group $\{\sigma_t^\psi\}$ of the faithful weight $\psi = \varphi_h$ given by (9) is of the form:*

$$\sigma_t^\psi(x) = h^{it}\sigma_t^\varphi(x)h^{-it}, \qquad x \in \mathcal{M},\ t \in \mathbb{R}.$$

Proof. Lemma 5.12 implies that ψ is a faithful weight. For each $n \in \mathbb{N}$, let e_n be the spectral projection of h corresponding to the interval $\left[\frac{1}{n}, n\right]$. Then the restriction of φ to $\mathcal{M}_{e_n} (= e_n\mathcal{M}e_n)$ is a faithful weight on $\mathcal{M}_{e_n}$ whose modular automorphism group is merely the restriction of $\{\sigma_t^\varphi\}$ to $\mathcal{M}_{e_n}$. Hence the previous lemma implies that if $x \in \mathcal{M}_{e_n}$, then

$$\sigma_t^\psi(x) = (he_n^{it})\sigma_t^\varphi(x)(he_n)^{-it} = h^{it}\sigma_t^\varphi(x)h^{-it}, \qquad t \in \mathbb{R}.$$

Hence this formula holds for all $x \in \bigcup_{n=1}^\infty \mathcal{M}_{e_n}$, which is σ-weakly dense in $\mathcal{M}$. □

6. The Connes Cocycle Derivative

The last few theorems of the previous section showed how perturbing a weight resulted in a simple relationship between the two modular automorphism groups, namely conjugating by a unitary group. The major result of this section (and in fact this could be considered the first major result in these notes) will show that given *any* two faithful, semi-finite, normal weights φ and ψ, their modular automorphism groups differ by conjugation by a unitary.

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